

Equations For Parseval's Frame Wavelets In $L^2(\mathbb{R}^d)$ With Compact Supports

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Abstract

Let $d \geq 1$ be a natural number and A_0 be a $d \times d$ expansive integral matrix with determinant ± 2 . Then A_0 is integrally similar to an integral matrix A with certain additional properties. A finite solution to the system of equations associated with the matrix A will result in an iterated sequence $\{\Psi^k \chi_{[0,1)^d}\}$ that converges to a function φ_A in $L^2(\mathbb{R}^d)$ -norm. With this (scaling) function φ_A , we will construct the Parseval's wavelet function ψ with compact support associated with matrix A_0 .

Keywords: Hilbert space $L^2(\mathbb{R}^d)$, expansive integral matrix, Parseval's frame wavelet, compact support.

1. Introduction

In this paper, d is a natural and $d \geq 1$, \mathbb{R} is the real numbers and \mathbb{C} is the complex numbers. \mathbb{R}^d will be the d -dimensional real Euclidean space and \mathbb{C}^d will be the d -dimensional complex Euclidean space. \mathbb{R}^d is a subset of \mathbb{C}^d . Let $\{\vec{e}_j, j = 1, \dots, d\}$ be the standard basis. For two vectors $\vec{x} = \sum_{j=1}^d x_j \vec{e}_j$ and $\vec{y} = \sum_{j=1}^d y_j \vec{e}_j$, the inner product of \vec{x} and \vec{y} is $\vec{x} \circ \vec{y} \equiv \sum_{j=1}^d x_j y_j$. For a vector $\vec{\xi} \in \mathbb{C}^d$, its real part $\Re(\vec{\xi})$ and imaginary part $\Im(\vec{\xi})$ are vectors in \mathbb{R}^d with $\vec{\xi} = \Re(\vec{\xi}) + i\Im(\vec{\xi})$. The measure μ will be the Lebesgue measure on \mathbb{R}^d and $L^2(\mathbb{R}^d)$ will be the Hilbert space of all square integrable functions on \mathbb{R}^d . A countable set of elements $\{\psi_i : i \in \Lambda\}$ in $L^2(\mathbb{R}^d)$ is called a *normalized tight frame* of $L^2(\mathbb{R}^d)$ if

$$(1.1) \quad \sum_{i \in \Lambda} |\langle f, \psi_i \rangle|^2 = \|f\|^2, \quad \forall f \in L^2(\mathbb{R}^d).$$

It is well known in the literature [4] that the Equation (1.1) is equivalent to

$$(1.2) \quad f = \sum_{i \in \Lambda} \langle f, \psi_i \rangle \psi_i, \quad \forall f \in L^2(\mathbb{R}^d),$$

where the convergence is in $L^2(\mathbb{R}^d)$ -norm. For a vector $\vec{\ell} \in \mathbb{R}^d$, the *translation operator* $T_{\vec{\ell}}$ is defined as

$$(T_{\vec{\ell}} f)(\vec{t}) \equiv f(\vec{t} - \vec{\ell}), \quad \forall f \in L^2(\mathbb{R}^d).$$

Let A be a $d \times d$ integral matrix with eigenvalues $\{\beta_1, \dots, \beta_d\}$. A is called *expansive* if $\min\{|\beta_j|, j \leq d\} > 1$. The norm of the linear transformation A on \mathbb{R}^d (or \mathbb{C}^d) will be $\|A\| \equiv \max\{|\beta_j|, j = 1, \dots, d\}$. For two vectors \vec{t}_1, \vec{t}_2 in the Euclidean space \mathbb{C}^d ,

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we have $\vec{t}_1 \circ A\vec{t}_2 = A^\tau \vec{t}_1 \circ \vec{t}_2$, where A^τ is the transpose matrix of the *real* matrix A . A^τ is expansive iff A is. We define operator U_A as

$$(U_A f)(\vec{t}) \equiv (\sqrt{|\det(A)|})f(A\vec{t}), \forall f \in L^2(\mathbb{R}^d).$$

This is a unitary operator. In particular, for an expansive matrix A with $|\det(A)| = 2$, we will use D_A for U_A and call it the *dilation operator associated with A* . The integral lattice \mathbb{Z}^d is an Abelian group under vector addition. The subset $(2\mathbb{Z})^d$ is a subgroup. For a fixed $d \times d$ integral matrix A with $|\det(A)| = 2$, the two sets $A\mathbb{Z}^d$ and $A^\tau \mathbb{Z}^d$ are proper subgroups of \mathbb{Z}^d containing $(2\mathbb{Z})^d$. An integral matrix B is said *integrally similar* to another integral matrix C by matrix S if $C = SBS^{-1}$ for an integral matrix S with S^{-1} being also integral.

DEFINITION 1.1. *Let A be an expansive integral matrix with $|\det(A)| = 2$. A function $\psi \in L^2(\mathbb{R}^d)$ is called a Parseval's frame wavelet associated with A , if the set*

$$(1.3) \quad \{D_A^n T_{\vec{\ell}} \psi, n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d\}$$

constitutes a normalized tight frame of $L^2(\mathbb{R}^d)$.

REMARK 1.1. *The function ψ is also called single function normalized tight frame wavelet. A Parseval's frame wavelet is not necessarily a unit vector in $L^2(\mathbb{R}^d)$ unless it is an orthonormal wavelet. By definition an element $\psi \in L^2(\mathbb{R}^d)$ is a Parseval's frame wavelet if and only if*

$$(1.4) \quad \|f\|^2 = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^n T_{\vec{\ell}} \psi \rangle|^2, \forall f \in L^2(\mathbb{R}^d).$$

By Equation (1.2) this is equivalent to

$$(1.5) \quad f = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d} \langle f, D_A^n T_{\vec{\ell}} \psi \rangle D_A^n T_{\vec{\ell}} \psi, \forall f \in L^2(\mathbb{R}^d).$$

In this paper, we will prove that an expansive integral matrix A_0 with determinant value ± 2 is integrally similar to an integral matrix A with certain additional properties. We will prove that a finite solution to the system of equations (6.1) associated with the matrix A will result in an iterated sequence $\{\Psi^k \chi_{[0,1)^d}\}$ that converges to a function φ_A . With this function φ_A , we will construct the Parseval's wavelet function ψ with compact support associated with matrix A_0 .

The original ideas of the above construction were due to W. Lawton on the one dimensional model [18] in 1990. People use the ideas, however, there is no paper clearly state and prove the above constructions in general d dimensional case, the generalization of Lawton's results into $L^2(\mathbb{R}^d)$. This paper give a clear formulation together with a rigorous proof.

In [5] we state and prove the above construction in the case $L^2(\mathbb{R}^2)$, the two dimensional case. It is natural that in some reasoning in this paper which is for general cases $d \geq 1$ we use the similar ideas for the corresponding conclusions in a previous paper [5]. In order to control the size of this paper and also to make this paper readable, we placed the proofs to some eleven statements into the Appendix. The 10 page Appendix includes proof of Theorem 5.1; proofs of Propositions 6.1, 8.2, 9.1, 9.2 and 9.3; proofs of Lemmas 3.2, 3.3, 8.1, 8.3 and 9.2. We have to note that the above mentioned proofs are different and improved somehow than corresponding parts in [5].

In the one dimensional case, the 1×1 dilation matrix 2 is simple: Its transpose is itself; it acts on the integer lattice \mathbb{Z} will result in an simple sublattice $2\mathbb{Z}$, the even integers; and the integer lattice \mathbb{Z} has a partition of even and odd numbers. However, for a $d \times d$ integral matrix A , these things have been changed. For example, $A\mathbb{Z}^d$ and $A^T\mathbb{Z}^d$ can be different sublattices of \mathbb{Z}^d . The desired property $A\mathbb{Z}^d = A^T\mathbb{Z}^d$ will make the situation relatively simpler when our discussions are in the frequency domain and the time domain at same time. We formulated the properties of matrices and related lattices we will need in higher dimensional cases into the Partition Theorem (Theorem 4.1). These properties have been cited in some key Propositions and their proofs (Propositions 6.1, 7.1, 9.1). In particular, we need these properties when we define the single function Parseval's frame wavelet (Definition 9.1). Sections 3 and 4 are devoted to prepare and prove this Partition Theorem.

The next fundamentally important issue is to show that the scaling function φ_A is in $L^2(\mathbb{R}^d)$. Due to the variety of the matrices we have to cover, we insert technical Lemmas (Lemmas 7.1, 7.2, 7.3) and Proposition 7.1 in Section 7. We will follow the classical method for constructing such frame wavelets as provided by I. Daubechies in [8]. That is, from the filter function m_0 to the scaling function φ . This is outlined in Section 2 Steps (C₁)-(C₃). Then we construct the wavelet function ψ . To construct the filter function m_0 we start with the system of equations (6.1) which is a direct generalization of W. Lawton's system of equations [18] for frame wavelets in $L^2(\mathbb{R})$. In Section 11, we provide two examples to show that the constructions we proved in this paper does produce Parseval's frame wavelets, even orthonormal wavelets.

The literature of wavelet theory in higher dimensions is rich. Many authors provide significant contributions to the theory. It is hard to make a short list. However, the author must cite the following names and their papers.

Q. Gu and D. Han [10] proved that, if an integral expansive matrix is associated with single function *orthogonal wavelets* with *multi-resolution analysis* (MRA), then the matrix determinant must be ± 2 . Orthogonal wavelets are special single function Parseval's frame wavelets.

The existence of Haar type orthonormal wavelets (hence with compact support) in $L^2(\mathbb{R}^2)$ was proved by J. Lagarias and Y. Wang in [17]. The first examples of such functions with compact support and with properties of high smoothness in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$ were provided by E. Belogay and Y. Wang in [2]. Compare with [17] and [2] our methods appear to be more constructive. Also, it provides variety for single function Parseval wavelets which includes the orthogonal wavelets. The methods in this paper provide a wide base in searching for more frame wavelets with normal properties as wavelets in [2].

The scaling function φ in this paper is not necessarily orthogonal. So the wavelet system constructed fits the definition of the *frame multi-resolution analysis* (FMRA) by J. Benedetto and S. Li in [3] and it also fits the definition of the *general multi-resolution analysis* (GMRA) by L. Baggett, H. Medina and K. Merrill in [1].

2. Main Results

Let A_0 be a $d \times d$ expansive integral matrix with $|\det(A_0)| = 2$. We will construct Parseval's frame wavelets associated with A_0 in the following steps (A)-(E).

(A) Find a $d \times d$ integral matrix A (Partition Theorem) which is integrally similar to A_0 with the following properties,

(1)

$$S^{-1}AS = A_0,$$

where S is an integral matrix with $|\det(S)| = 1$.

(2)

$$AZ^d = A^T Z^d.$$

(3) There exists a vector $\vec{\ell}_A \in \mathbb{Z}^d$ such that

$$\mathbb{Z}^d = (\vec{\ell}_A + AZ^d) \cup AZ^d.$$

(4) There exists a vector $\vec{q}_A \in \mathbb{Z}^d$

$$\vec{q}_A \circ AZ^d \subseteq 2\mathbb{Z} \quad \text{and} \quad \vec{q}_A \circ (\vec{\ell}_A + AZ^d) \subseteq 2\mathbb{Z} + 1.$$

(B) Solve the system of equations

$$\begin{cases} \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} \overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}, \quad \vec{k} \in AZ^d \\ \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} = \sqrt{2}. \end{cases}$$

for a finite solution $\mathcal{S} = \{h_{\vec{n}} : \vec{n} \in \mathbb{Z}^d\}$. We say \mathcal{S} is a finite solution if the index set of non-zero terms $h_{\vec{n}}$ is included in the set $\Lambda_0 \equiv \mathbb{Z}^d \cap [-N_0, N_0]^d$ for some natural number N_0 .

(C) Let Ψ be the linear operator on $L^2(\mathbb{R}^d)$ by

$$\Psi \equiv \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_A T_{\vec{n}}.$$

The iterated sequence $\{\Psi^k \chi_{[0,1)^d}, k \in \mathbb{N}\}$ will converge to the scaling function φ_A in the $L^2(\mathbb{R}^d)$ -norm (Theorem 10.2).

(D) Define function ψ_A

$$\psi_A \equiv \sum_{\vec{n} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} D_A T_{\vec{n}} \varphi_A.$$

This is a Parseval's frame wavelet with compact support associated with matrix A (Theorem 9.1).

(E) Define the wavelet function ψ by

$$\psi(\vec{t}) \equiv \psi_A(S\vec{t}), \forall \vec{t} \in \mathbb{R}^d.$$

The function ψ is a Parseval's frame wavelet with compact support associated with the given matrix A_0 (Theorem 5.1).

To prove the conclusion in step (C) and that φ_A has a compact support, we do the following steps (C₁)-(C₃).

(C₁) Define the filter function $m_0(\vec{t})$.

$$m_0(\vec{t}) \equiv \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} e^{-i\vec{n} \cdot \vec{t}}.$$

(C₂) Define a function g by

$$g(\vec{\xi}) \equiv \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^{\infty} m_0((A^T)^{-j} \vec{\xi}), \forall \vec{\xi} \in \mathbb{R}^d$$

The function g is an $L^2(\mathbb{R}^d)$ -function (Proposition 7.1), and its extension is an entire function on \mathbb{C}^d (Proposition 8.1).

(C₃) Define the scaling function φ_A

$$\varphi_A \equiv \mathcal{F}^{-1} g,$$

where \mathcal{F} is the Fourier transform. The scaling function φ_A is an $L^2(\mathbb{R}^d)$ -function with compact support (Proposition 8.2).

3. Factorization of Matrices

In this section, we will prove a factorization theorem (Proposition 3.1) of integral matrices we discuss. This will provide a base to prove the Partition Theorem in Section 4. Let \mathfrak{M} be the set of $d \times d$ integral matrices with determinants ± 1 . This is a group under matrix multiplication. We will use some elementary $d \times d$ integral matrices. We state their properties in Lemma 3.1.

Let p, i, j be natural numbers and $1 \leq p, i, j \leq d$. Let I be the $d \times d$ identity matrix. Denote Δ_{ij} the $d \times d$ matrix with 1 at ij position and 0 at all other positions. Denote I_{ij} be the matrix after interchanging the i^{th} and the j^{th} columns in the identity matrix I . Denote $D_p \equiv I + \Delta_{pp}$, and $S_p \equiv I - 2\Delta_{pp}$.

LEMMA 3.1. *Let B be a $d \times d$ matrix, and m, n, p, k, ℓ, i, j be natural numbers in $[1, d]$.*

(1)

$$(3.1) \quad \Delta_{mn} \cdot \Delta_{k\ell} = \delta_{nk} \Delta_{m\ell}$$

where δ is the Kronecker delta.

(2) *If $i \neq j$, then*

$$(3.2) \quad (I + \Delta_{ij})^{-1} = I - \Delta_{ij}.$$

Right multiply $(I \pm \Delta_{ij})$ to B will add the i^{th} column to the j^{th} in matrix B (subtract the i^{th} column from the j^{th} column in matrix B).

(3) $S_p = S_p^{-1}$. *Right multiply S_p to B will change the sign of the p^{th} column.*

(4) $I_{ij} = I_{ji} = I_{ij}^{-1}$. *Right multiply I_{ij} to B will interchange the i^{th} and the j^{th} columns in matrix B .*

(5) $D_p^{-1} = I - \frac{1}{2}\Delta_{pp} \notin \mathfrak{M}$. *Right multiply D_p to B will multiply the p^{th} column in B by 2.*

An integral matrix B is said *integrally similar* to another integral matrix C by matrix S if $C = SBS^{-1}$ for an integral matrix S with S^{-1} being also integral. The only possible integral matrices to serve for S are matrices in the group \mathfrak{M} . We will call the matrices in the set

$$\mathfrak{G} = \{S_p\} \cup \{(I \pm \Delta_{ij}), i \neq j\} \cup \{I_{ij}\}$$

elementary matrices. Every member in \mathfrak{G} has its inverse also in \mathfrak{G} . Also, we have $\mathfrak{G} \subset \mathfrak{M}$.

LEMMA 3.2. *Let B be a $d \times d$ non singular integral matrix. Then B can be factored into*

$$B = LV$$

where $V \in \mathfrak{M}$ which is the finite product of elements from \mathfrak{G} and L an integral lower triangular matrix with positive diagonals $\{\ell_{11}, \dots, \ell_{dd}\}$.

(For the proof, see Appendix PROOF OF LEMMA 3.2.)

LEMMA 3.3. *An integral matrix in \mathfrak{M} is the product of finite elements from \mathfrak{G} .*

(For the proof, see Appendix PROOF OF LEMMA 3.3.)

Let \vec{r} be a d -dimensional row vector for which the only possible non-zero coordinates are the first $p-1$ entries with values 1 or 0,

$$\vec{r} \equiv [r_1, r_2, \dots, r_{p-1}, 0, \dots, 0], r_j \in \{0, 1\}.$$

Define matrix $L_{\vec{r}}^{(p)}$ and $M_{\vec{r}}^{(p)}$ by

$$\begin{aligned} L_{\vec{r}}^{(p)} &\equiv I + \sum_{j=1}^{p-1} r_j \Delta_{pj}, p \leq d; \\ M_{\vec{r}}^{(p)} &\equiv I + \sum_{j=1}^{p-1} r_j \Delta_{dj}, p \leq d. \end{aligned}$$

We have

$$\begin{aligned} (L_{\vec{r}}^{(p)})^{-1} &= I - \sum_{j=1}^{p-1} r_j \Delta_{pj}; \\ I_{pd} L_{\vec{r}}^{(p)} I_{pd} &= M_{\vec{r}}^{(p)}. \end{aligned}$$

PROPOSITION 3.1. *Let B be an integral matrix with $|\det(B)| = 2$. Then for some natural number $p \leq d, n_0$ and m_0 , and some vector*

$$\vec{r} \equiv [r_1, r_2, \dots, r_{p-1}, 0, \dots, 0], r_j \in \{0, 1\}.$$

we have

$$(3.3) \quad B = L_{\vec{r}}^{(p)} D_p V.$$

$$(3.4) \quad B = I_{pd} M_{\vec{r}}^{(p)} D_d U.$$

$$(3.5) \quad B = V_1 V_2 \dots V_{n_0} D_d U_1 U_2 \dots U_{m_0}.$$

where V and U are matrices in \mathfrak{M} , $V_j, U_i \in \mathfrak{G}$, $L_{\vec{r}}^{(p)}$ and $M_{\vec{r}}^{(p)}$ are as defined as before Proposition 3.1

PROOF. Let B be an integral matrix with $\det B = \pm 2$. By Lemma 3.2, we have $B = LU$ where $L = (\ell_{ij})$ is lower triangular and $U \in \mathfrak{M}$. So $\ell_{ij} = 0$, if $i < j$ and $\ell_{ii} > 0, i = 1, \dots, d$, and $\det(L) = \prod \ell_{jj} = 2$. Therefore, every diagonal element has value 1 except for $\ell_{pp} = 2$ for some $p \leq d$.

By the methods we used in proof of Lemma 3.3, (we do all right multiplication but $i = p$.) we can factor $B = LU = L_1 V_1$ where V_1 is the product of some finite elements in \mathfrak{G} and $L_1 = (c_{ij})$ is lower triangular, and its p^{th} -row is $[c_{p1} \ c_{p2} \ \dots \ c_{p(p-1)} \ 2 \ 0 \ \dots \ 0]$ and all entries in other rows are the same entries in corresponding positions as in identity matrix I . For $j = 1, \dots, p-1$, $c_{pj} = 2m_j + r_j, 0 \leq r_j \leq 1$. We define the row vector \vec{r}

$$\vec{r} \equiv [r_{p1} \ r_{p2} \ \dots \ r_{p(p-1)} \ 2 \ 0 \ \dots \ 0].$$

Notice that the inverse to $(I - \Delta_{pj})^{m_j}$ is $(I + \Delta_{pj})^{m_j}$, we have

$$\begin{aligned} B &= L_1 V_1 = (L_1 (I - \Delta_{p(p-1)})^{m_{(p-1)}}) ((I + \Delta_{p(p-1)})^{m_{(p-1)}} V_1) \\ &= (L_1 (I - \Delta_{p(p-1)})^{m_{(p-1)}} \dots (I - \Delta_{p2})^{m_2}) \cdot \\ &\quad ((I + \Delta_{p2})^{m_2} \dots (I + \Delta_{p(p-1)})^{m_{(p-1)}} V_1) \\ &= L_0 V, \text{ where} \\ L_0 &\equiv L_1 (I - \Delta_{p(p-1)})^{m_{(p-1)}} \dots (I - \Delta_{p2})^{m_2}, \text{ and} \\ V &\equiv (I + \Delta_{p2})^{m_2} \dots (I + \Delta_{p(p-1)})^{m_{(p-1)}} V_1. \end{aligned}$$

It is clear that V is the product of finite elements from \mathfrak{G} and it is in \mathfrak{M} . Also, for $j < p$, the only difference between matrices L_1 and $L_1(I - \Delta_{p2})^{m_2}$ is, at pj position the first matrix has value c_{pj} and the second (product) matrix has value r_j . So L_0 is a lower triangular matrix for which if $j \neq p$ then the j^{th} row vector is \vec{e}_j and the p^{th} row vector is $\vec{r} + 2\vec{e}_p$. So, $L_0 = L_{\vec{r}}^{(p)} D_p$. Hence we have $B = L_{\vec{r}}^{(p)} D_p V$. Since $I_{dp}^2 = I$, we have

$$B = L_{\vec{r}}^{(p)} D_p V = I_{dp} (I_{dp} L_{\vec{r}}^{(p)} I_{dp}) (I_{dp} D_p I_{dp}) (I_{dp} V) = I_{dp} M_{\vec{r}}^{(p)} D_d U.$$

Here $U \equiv I_{dp} V$ is the finite product of elements from \mathfrak{G} . Its is easy to verify that $I_{dp} D_p I_{dp} = D_d$ and $I_{dp} L_{\vec{r}}^{(p)} I_{dp} = M_{\vec{r}}^{(p)}$.

□

4. Partition Theorem

The purpose for this section is to establish the following Theorem 4.1. We call it Partition Theorem. The partition properties are essential in our approach in this paper.

THEOREM 4.1. *Every integral matrix B with $|\det(B)| = 2$ is integrally similar to an integral matrix A with the properties that*

(1)

$$(4.1) \quad A\mathbb{Z}^d = A^T \mathbb{Z}^d.$$

(2) *There exists a vector $\vec{\ell}_A \in \mathbb{Z}^d$ such that*

$$(4.2) \quad \mathbb{Z}^d = (\vec{\ell}_A + A\mathbb{Z}^d) \cup A\mathbb{Z}^d.$$

(3) There exists a vector $\vec{q}_A \in \mathbb{Z}^d$

$$(4.3) \quad \vec{q}_A \circ A\mathbb{Z}^d \subseteq 2\mathbb{Z} \quad \text{and} \quad \vec{q}_A \circ (\vec{\ell}_A + A\mathbb{Z}^d) \subseteq 2\mathbb{Z} + 1.$$

(4) For $\vec{m} \in A\mathbb{Z}^d$, we have

$$(4.4) \quad \mathbb{Z}^d = (\vec{n} - A\mathbb{Z}^d) \cup (\vec{\ell}_A - \vec{m} - \vec{n} + A\mathbb{Z}^d), \quad \forall \vec{n} \in \mathbb{Z}^d.$$

(5) For $\vec{m} \in \vec{\ell}_A + A\mathbb{Z}^d$, we have

$$(4.5) \quad \vec{n} - A\mathbb{Z}^d = \vec{\ell}_A - \vec{m} - \vec{n} + A\mathbb{Z}^d, \quad \forall \vec{n} \in \mathbb{Z}^d.$$

REMARK 4.1. Note that the matrices in Theorem 4.1 are not necessarily expansive. For example, for $d \geq 2$, D_d is not expansive but determinant value is 2.

PROOF. By Proposition 3.1 we have

$$B = I_{pd} M_{\vec{r}}^{(p)} D_d U_1$$

for some $U_1 \in \mathfrak{M}$, $p \leq d$ and $\vec{r} = [r_1, r_2, \dots, r_{p-1}, 0, \dots, 0]$, $r_j \in \{0, 1\}$. Define

$$C_1 \equiv U_1 B U_1^{-1} = U_1 I_{pd} M_{\vec{r}}^{(p)} D_d.$$

We have $|\det(C_1)| = 2$. By Proposition 3.1 again,

$$C_1 = I_{qd} M_{\vec{s}}^{(q)} D_d U_2$$

for some $U_2 \in \mathfrak{M}$, $q \leq d$ and $\vec{s} = [s_1, s_2, \dots, s_{q-1}, 0, \dots, 0]$, $s_j \in \{0, 1\}$. Define $C_2 \equiv (I_{qd} M_{\vec{s}}^{(q)})^{-1} C_1 (I_{qd} M_{\vec{s}}^{(q)})$. We have

$$\begin{aligned} C_2 &= D_d U_2 (I_{qd} M_{\vec{s}}^{(q)}) \\ &= D_d W_1 \end{aligned}$$

where $W_1 \equiv U_2 (I_{qd} M_{\vec{s}}^{(q)}) \in \mathfrak{M}$. Also we have

$$\begin{aligned} C_2 &= (I_{qd} M_{\vec{s}}^{(q)})^{-1} U_1 I_{pd} M_{\vec{r}}^{(p)} D_d (I_{qd} M_{\vec{s}}^{(q)}) \\ &= W_2 D_d I_{qd} M_{\vec{s}}^{(q)} \end{aligned}$$

where $W_2 \equiv (I_{qd} M_{\vec{s}}^{(q)})^{-1} U_1 I_{pd} M_{\vec{r}}^{(p)} \in \mathfrak{M}$. Define a matrix X be

$$(4.6) \quad X \equiv \begin{cases} I & \text{if } q = d, \\ I - \Delta_{dq} & \text{if } q < d. \end{cases}$$

Define $A \equiv X C_2 X^{-1}$. It is clear that the matrices A and B are integrally similar. We claim that the matrix A has the desired properties (1), (2), (3), (4) and (5).

We have

$$\begin{aligned} A &= X D_d W_1 X^{-1}, \\ A &= X W_2 D_d I_{qd} M_{\vec{s}}^{(q)} X^{-1}, \end{aligned}$$

then we have

$$(4.7) \quad A^\tau = (X^{-1})^\tau (M_{\vec{s}}^{(q)})^\tau I_{qd} D_d (W_2)^\tau X^\tau,$$

and

$$(4.8) \quad A^{-1} = X (W_1)^{-1} (D_d)^{-1} X^{-1}.$$

Proof of Property (1). It is clear that $|\det(A^{-1}A^\tau)| = 1$. We will show that the product $A^{-1}A^\tau$ is an integral matrix. Then we will have $A^{-1}A^\tau\mathbb{Z}^d = \mathbb{Z}^d$. This is equivalent to the equation $A^\tau\mathbb{Z}^d = A\mathbb{Z}^d$. We have two cases.

Case (A) First we assume $q < d$. By (4.8) and (4.7) we have

$$A^{-1}A^\tau = X(W_1)^{-1}[(D_d)^{-1}X^{-1}(X^{-1})^\tau(M_s^{(q)})^\tau I_{qd}D_d](W_2)^\tau X^\tau$$

$(D_d)^{-1}$ and D_d are the only matrices on the right product which are not in \mathfrak{M} . We will prove that the product $(D_d)^{-1}X^{-1}(X^{-1})^\tau(M_s^{(q)})^\tau I_{qd}D_d$ is an integral matrix. This will complete the proof of (1).

It is clear that $X^{-1} = I + \Delta_{dq}$ and $(X^{-1})^\tau = I + \Delta_{qd}$ and $(D_d)^{-1} = I - \frac{1}{2}\Delta_{dd}$. So we have

$$\begin{aligned} (D_d)^{-1}X^{-1} \cdot (X^{-1})^\tau &= (I - \frac{1}{2}\Delta_{dd}) \cdot (I + \Delta_{dq}) \cdot (I + \Delta_{qd}) \\ &= (I - \frac{1}{2}\Delta_{dd}) \cdot (I + \Delta_{qd} + \Delta_{dq} + \Delta_{dd}) \\ &= I + \Delta_{qd} + \frac{1}{2}\Delta_{dq}. \end{aligned}$$

Here we used the Equation (3.1). We also have

$$\begin{aligned} (M_s^{(q)})^\tau I_{qd} &= (I + \sum_{j=1}^{q-1} s_j \Delta_{dj})^\tau \cdot I_{qd} \\ &= (I + \sum_{j=1}^{q-1} s_j \Delta_{jd}) \cdot I_{qd} \\ &= I_{qd} + \sum_{j=1}^{q-1} s_j \Delta_{jq}. \end{aligned}$$

Repeatedly using Equation (3.1), we obtain that

$$\begin{aligned} A^{-1}A^\tau &= ((D_d)^{-1}X^{-1} \cdot (X^{-1})^\tau) \cdot ((M_s^{(q)})^\tau I_{qd}) \cdot D_d \\ &= (I + \Delta_{qd} + \frac{1}{2}\Delta_{dq}) \cdot (I_{qd} + \sum_{j=1}^{q-1} s_j \Delta_{jq}) \cdot D_d \\ &= (\frac{1}{2}\Delta_{dd} + I_{qd} + \sum_{j=1}^{q-1} s_j \Delta_{jq}) \cdot D_d \\ &= \Delta_{dd} + 2I_{qd} + \sum_{j=1}^{q-1} s_j \Delta_{jq}. \end{aligned}$$

This is an integral matrix. In the above last step we used the following equality

$$(4.9) \quad \Delta_{ij} \cdot D_d = \begin{cases} 2\Delta_{ij} & j = d, \\ \Delta_{ij} & j < d. \end{cases}$$

Case (B) Next, we assume $q = d$. By definition (4.6) $X = I$. We have $A = C_2 = D_d W_1 = W_2 D_d M_s^{(d)}$. So, $A^{-1} = (W_1)^{-1}(D_d)^{-1}$ and $A^\tau = (M_s^{(d)})^\tau D_d (W_2)^\tau$.

$$A^{-1}A^\tau = W_1^{-1}(D_d)^{-1}(M_s^{(d)})^\tau D_d W_2^\tau.$$

Since W_1^{-1} and W_2^τ are integral matrices, it is enough to prove that the product $(D_d)^{-1}(M_s^{(d)})^\tau D_d$ is integral.

$$\begin{aligned}
(D_d)^{-1}(M_s^{(d)})^\tau D_d &= (I - \frac{1}{2}\Delta_{dd}) \cdot (I + \sum_{j=1}^{d-1} s_j \Delta_{dj})^\tau \cdot D_d \\
&= (I - \frac{1}{2}\Delta_{dd}) \cdot (I + \sum_{j=1}^{d-1} s_j \Delta_{jd}) \cdot D_d \\
&= (I - \frac{1}{2}\Delta_{dd} + \sum_{j=1}^{d-1} s_j \Delta_{jd}) \cdot D_d \\
&= I + \sum_{j=1}^{d-1} 2s_j \Delta_{jd}.
\end{aligned}$$

So, $A^{-1}A^\tau$ an integral matrix. Proof of Property (1) is complete.

Proof of Property (2). Since $|\det(A)| = 2$, by Proposition 3.1 we have

$$(4.10) \quad A = I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d U_0$$

for some $p_0 \leq d$, $U_0 \in \mathfrak{M}$ and vector $\vec{r}_0 \equiv \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j$, $\varrho_j \in \{0, 1\}$. This is a vector in \mathbb{Z}^d . Also we have

$$(4.11) \quad M_{\vec{r}_0}^{(p_0)} = I + \sum_{j=1}^{p_0-1} \varrho_j \Delta_{dj}.$$

Let $\vec{v} = \sum_j x_j \vec{e}_j$ be a vector in \mathbb{Z}^d . The coefficient x_d is even or odd. So we have

$$\mathbb{Z}^d = (\vec{e}_d + D_d \mathbb{Z}^d) \cup D_d \mathbb{Z}^d.$$

All matrices $I_{p_0 d}$, $M_{\vec{r}_0}^{(p_0)}$ and U_0 are in \mathfrak{M} . So $U_0 \mathbb{Z}^d = \mathbb{Z}^d$ and $I_{p_0 d} M_{\vec{r}_0}^{(p_0)} \mathbb{Z}^d = \mathbb{Z}^d$. We have

$$\begin{aligned}
\mathbb{Z}^d &= I_{p_0 d} M_{\vec{r}_0}^{(p_0)} \mathbb{Z}^d \\
&= I_{p_0 d} M_{\vec{r}_0}^{(p_0)} ((\vec{e}_d + D_d \mathbb{Z}^d) \cup D_d \mathbb{Z}^d) \\
&= I_{p_0 d} M_{\vec{r}_0}^{(p_0)} ((\vec{e}_d + D_d U_0 \mathbb{Z}^d) \cup D_d U_0 \mathbb{Z}^d)
\end{aligned}$$

Define

$$\vec{\ell}_A \equiv I_{p_0 d} M_{\vec{r}_0}^{(p_0)} \vec{e}_d = \vec{e}_{p_0}.$$

By Equation (4.10) we obtain

$$\begin{aligned}
\mathbb{Z}^d &= (I_{p_0 d} M_{\vec{r}_0}^{(p_0)} \vec{e}_d + I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d U_0 \mathbb{Z}^d) \cup I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d U_0 \mathbb{Z}^d \\
&= (\vec{\ell}_A + A \mathbb{Z}^d) \cup A \mathbb{Z}^d.
\end{aligned}$$

Proof of Property (3). Case (A). Assume $p_0 < d$. We will say that $A \mathbb{Z}^d$ is integrally generated by a (finite) set $\{\vec{u}_j\}$ if for each vector $\vec{v} \in A \mathbb{Z}^d$ there exists a (finite) set of integers $\{s_j\}$ such that $\vec{v} = \sum s_j \vec{u}_j$. We will call the set a generator for $A \mathbb{Z}^d$. The set $\{A \vec{e}_j, j = 1, \dots, d\}$ is a generator for $A \mathbb{Z}^d$. By Equation (4.10)

$A = I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d U_0$. Since $U_0 \mathbb{Z}^d = \mathbb{Z}^d$, the set $\{I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d \vec{e}_i, i = 1, \dots, d\}$ is a generator for $A \mathbb{Z}^d$. We have

$$\begin{aligned}
 I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d \vec{e}_i &= I_{p_0 d} \left(I + \sum_{j=1}^{p_0-1} \varrho_j \Delta_{dj} \right) D_d \vec{e}_i \\
 &= \begin{cases} I_{p_0 d} ((1 + \varrho_i) \vec{e}_i) & \text{if } i \leq p_0 - 1, \\ I_{p_0 d} (\vec{e}_{p_0}) & \text{if } i = p_0, \\ I_{p_0 d} (\vec{e}_i) & \text{if } p_0 + 1 \leq i \leq d - 1, \\ I_{p_0 d} (2\vec{e}_d) & \text{if } i = d. \end{cases} \\
 &= \begin{cases} (1 + \varrho_i) \vec{e}_i & \text{if } i \leq p_0 - 1, \\ \vec{e}_d & \text{if } i = p_0, \\ \vec{e}_i & \text{if } p_0 + 1 \leq i \leq d - 1, \\ 2\vec{e}_{p_0} & \text{if } i = d. \end{cases}
 \end{aligned}$$

Define

$$(4.12) \quad \vec{q}_A \equiv \vec{e}_{p_0} + \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j.$$

To prove that $\vec{q}_A \circ A \mathbb{Z}^d \subseteq 2\mathbb{Z}$, it is enough to show that $\vec{q}_A \circ (I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d \vec{e}_i)$ is an even integer for each $i \leq d$. We have

$$\begin{aligned}
 &\vec{q}_A \circ (I_{p_0 d} M_{\vec{r}_0}^{(p_0)} D_d \vec{e}_i) \\
 &= \begin{cases} \vec{q}_A \circ ((1 + \varrho_i) \vec{e}_i) & \text{if } i \leq p_0 - 1, \\ \vec{q}_A \circ \vec{e}_d & \text{if } i = p_0, \\ \vec{q}_A \circ \vec{e}_i & \text{if } p_0 + 1 \leq i \leq d - 1, \\ \vec{q}_A \circ (2\vec{e}_{p_0}) & \text{if } i = d. \end{cases} \\
 &= \begin{cases} (\vec{e}_{p_0} + \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j) \circ ((1 + \varrho_i) \vec{e}_i) & \text{if } i \leq p_0 - 1, \\ (\vec{e}_{p_0} + \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j) \circ \vec{e}_d & \text{if } i = p_0, \\ (\vec{e}_{p_0} + \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j) \circ \vec{e}_i & \text{if } p_0 + 1 \leq i \leq d - 1, \\ (\vec{e}_{p_0} + \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j) \circ (2\vec{e}_{p_0}) & \text{if } i = d. \end{cases} \\
 &= \begin{cases} \varrho_i (1 + \varrho_i) & \text{if } i \leq p_0 - 1, \\ 0 & \text{if } i = p_0, \\ 0 & \text{if } p_0 + 1 \leq i \leq d - 1, \\ 2 & \text{if } i = d. \end{cases}
 \end{aligned}$$

The values of all above inner products are even integers. So we proved that when $p_0 < d$, $\vec{q}_A \circ A \mathbb{Z}^d$ is a set of even integers. Also,

$$\vec{q}_A \circ \vec{\ell}_A = (\vec{e}_{p_0} + \sum_{j=1}^{p_0-1} \varrho_j \vec{e}_j) \circ \vec{e}_{p_0} = 1.$$

This implies that $\vec{q}_A \circ (\vec{\ell}_A + A \mathbb{Z}^d)$ is a set of odd integers.

Case (B) Now we assume $p_0 = d$. Then $I_{p_0 d} = I$. In this case

$$A = M_{\vec{r}_0}^{(d)} D_d U_0,$$

$$\vec{r}_0 = \sum_{j=1}^{d-1} \varrho_j \vec{e}_j, \varrho_j \in \{0, 1\}.$$

By the definition in Proof of Property (2)

$$\vec{\ell}_A = \vec{e}_d.$$

Define

$$\vec{q}_A \equiv \vec{e}_d + \sum_{j=1}^{d-1} \varrho_j \vec{e}_j.$$

It is clear that $\vec{q}_A \circ \vec{\ell}_A = (\vec{e}_d + \sum_{j=1}^{d-1} \varrho_j \vec{e}_j) \circ \vec{e}_d = 1$. Denote $\vec{u}_i \equiv M_{\vec{r}_0}^{(d)} D_d \vec{e}_i, i = 1, \dots, d$. Then the set $\{\vec{u}_i, i = 1, \dots, d\}$ is a generator for $A\mathbb{Z}^d$. To complete the proof of Case (B), it is enough to show that $\vec{q}_A \circ \vec{u}_i$ is even integer for each $i \leq d$. We have

$$\begin{aligned} \vec{u}_i &= M_{\vec{r}_0}^{(d)} D_d \vec{e}_i, i = 1, \dots, d \\ &= \begin{cases} M_{\vec{r}_0}^{(d)} \vec{e}_i & \text{if } i \leq d-1, \\ 2M_{\vec{r}_0}^{(d)} \vec{e}_d & \text{if } i = d. \end{cases} \\ &= \begin{cases} (1 + \varrho_i) \vec{e}_i & \text{if } i \leq d-1, \\ 2\vec{e}_d & \text{if } i = d. \end{cases} \end{aligned}$$

We have

$$\begin{aligned} \vec{q}_A \circ \vec{u}_i &= \begin{cases} (\vec{e}_d + \sum_{j=1}^{d-1} \varrho_j \vec{e}_j) \circ ((1 + \varrho_i) \vec{e}_i) & \text{if } i \leq d-1, \\ (\vec{e}_d + \sum_{j=1}^{d-1} \varrho_j \vec{e}_j) \circ (2\vec{e}_d) & \text{if } i = d. \end{cases} \\ &= \begin{cases} (1 + \varrho_i) \varrho_i & \text{if } i \leq d-1, \\ 2 & \text{if } i = d. \end{cases} \end{aligned}$$

Apparently, those are even numbers. Case (B) has been proved.

Proof of Property (4). Let $\vec{m} \in A\mathbb{Z}^d$. Since $A\mathbb{Z}^d$ is a group containing $(2\mathbb{Z})^d$, we have $-A\mathbb{Z}^d = A\mathbb{Z}^d$ and $-\vec{m} - \vec{n} + A\mathbb{Z}^d = \vec{n} + A\mathbb{Z}^d$. So

$$(4.13) \quad (\vec{n} - A\mathbb{Z}^d) \cup (\vec{\ell}_A - \vec{m} - \vec{n} + A\mathbb{Z}^d) = (\vec{n} + A\mathbb{Z}^d) \cup (\vec{\ell}_A + \vec{n} + A\mathbb{Z}^d).$$

If $\vec{n} \in A\mathbb{Z}^d$ then $\vec{n} + A\mathbb{Z}^d = A\mathbb{Z}^d$ and $\vec{\ell}_A + \vec{n} + A\mathbb{Z}^d = \vec{\ell}_A + A\mathbb{Z}^d$. If $\vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d$ then $\vec{n} + A\mathbb{Z}^d = \vec{\ell}_A + A\mathbb{Z}^d$ and $\vec{\ell}_A + \vec{n} + A\mathbb{Z}^d = A\mathbb{Z}^d$. So, in either cases we have

$$(\vec{n} + A\mathbb{Z}^d) \cup (\vec{\ell}_A + \vec{n} + A\mathbb{Z}^d) = (A\mathbb{Z}^d) \cup (\vec{\ell}_A + A\mathbb{Z}^d) = \mathbb{Z}^d.$$

Proof of Property (5) Let $\vec{m} \in \vec{\ell}_A + A\mathbb{Z}^d$. Then $\vec{\ell}_A - \vec{m} \in A\mathbb{Z}^d = A\mathbb{Z}^d$. So $\vec{\ell}_A - \vec{m} - \vec{n} + A\mathbb{Z}^d = \vec{n} - A\mathbb{Z}^d$. This is (4.5). \square

5. Reduction Theorem

For $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, the Fourier-Plancherel Transform and Fourier Inverse Transform are defined as

$$\begin{aligned} (\mathcal{F}f)(\vec{s}) &\equiv \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\vec{s} \circ \vec{t}} f(\vec{t}) d\vec{t} = \hat{f}(\vec{s}), \\ (\mathcal{F}^{-1}g)(\vec{t}) &\equiv \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\vec{s} \circ \vec{t}} g(\vec{s}) d\vec{s} = \check{g}(\vec{t}). \end{aligned}$$

The set $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the operator \mathcal{F} extends to a unitary operator on $L^2(\mathbb{R}^d)$ which is still called Fourier Transform. For an operator V on $L^2(\mathbb{R}^d)$, we will use notation $\widehat{V} \equiv \mathcal{F}V\mathcal{F}^{-1}$. We collect the following elementary equalities in Lemma 5.1. We omit the proof.

LEMMA 5.1. *Let A be a $d \times d$ expansive integral matrix with $|\det(A)| = 2$; \vec{t}, \vec{s} and $\vec{\ell} \in \mathbb{R}^d$ and $J \in \mathbb{Z}$. For a $d \times d$ integral matrix S of $|\det(S)| = 1$, assume $B = S^{-1}AS$. Then*

$$(5.1) \quad U_S T_{\vec{\ell}} U_S^{-1} = T_{S^{-1}\vec{\ell}}.$$

$$(5.2) \quad U_S D_A^J U_S^{-1} = D_B^J.$$

$$(5.3) \quad T_{\vec{\ell}} D_A = D_A T_{A\vec{\ell}}.$$

$$(5.4) \quad \widehat{T}_{\vec{\ell}} = M_{e^{-i\vec{s} \circ \vec{\ell}}}.$$

$$(5.5) \quad \widehat{D}_A = U_{(A^{-1})^\tau} = U_{(A^\tau)^{-1}} = D_{A^\tau}^{-1} = D_{A^\tau}^*.$$

$$(5.6) \quad T_{A^{-J}\vec{\ell}} D_A^J = D_A^J T_{\vec{\ell}}.$$

$$(5.7) \quad \overline{\widehat{D}_A^J \widehat{\phi}(\vec{t})} = \frac{1}{\sqrt{2^J}} \widehat{\phi}((A^\tau)^{-J}\vec{t}), \phi \in L^2(\mathbb{R}^d).$$

where $M_{e^{-i\vec{s} \circ \vec{\ell}}}$ is the unitary multiplication operator by $e^{-i\vec{s} \circ \vec{\ell}}$.

THEOREM 5.1. *Let A be a $d \times d$ expansive integral matrix with $|\det(A)| = 2$ and S be a $d \times d$ integral matrix with property that $|\det(S)| = 1$. Let $B \equiv S^{-1}AS$. Assume that a function ψ_A is a Parseval's frame wavelet associated with the matrix A . Then the function $\eta_B \equiv U_S \psi_A$ is a Parseval's frame wavelet associated with the matrix B .*

(For the proof, see Appendix PROOF OF THEOREM 5.1).

6. Lawton's Equations and Filter Function $m_0(\vec{t})$

Through out the rest of this paper, A will be a $d \times d$ expansive integral matrix with $|\det(A)| = 2$ satisfying the equations (4.1), (4.2), (4.3), (4.4) and (4.5) in Partition Theorem. Let $\mathcal{S} = \{h_{\vec{n}} : \vec{n} \in \mathbb{Z}^d\}$ be a *finite* complex solution to the system of equations

$$(6.1) \quad \begin{cases} \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} \overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}, \vec{k} \in A\mathbb{Z}^d \\ \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} = \sqrt{2}. \end{cases}$$

Here δ is the Kronecker delta. The solution is finite if $h_{\vec{n}} = 0$ for all $\vec{n} \notin \Lambda_0 \equiv \mathbb{Z}^d \cap [-N_0, N_0]^d$ for some $N_0 \in \mathbb{N}$. Wayne Lawton introduced [18] a system of equations for Parseval's frame wavelet in $L^2(\mathbb{R})$. The equations (6.1) are its generalization in higher dimensions. We will call the system of equations (6.1) *Lawton's system of equations for Parseval's frame wavelets associated with matrix A* , or *Lawton's equations associated with matrix A* .

Define

$$(6.2) \quad m_0(\vec{t}) \equiv \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} e^{-i\vec{n} \circ \vec{t}} = \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} e^{-i\vec{n} \circ \vec{t}}, \vec{t} \in \mathbb{C}^d.$$

This is a finite sum and $m_0(\vec{0}) = 1$. It is a 2π -periodic trigonometric polynomial function in the sense that $m_0(\vec{t}) = m_0(\vec{t} + \pi\vec{t}_0)$, $\forall \vec{t}_0 \in (2\mathbb{Z})^d$.

PROPOSITION 6.1. *Let A be an expansive $d \times d$ integral matrix and \vec{q}_A is as stated in Partition Theorem. Let $m_0(\vec{t})$ be defined as in (6.2), then*

$$(6.3) \quad |m_0(\vec{t})|^2 + |m_0(\vec{t} + \pi\vec{q}_A)|^2 = 1, \quad \forall \vec{t} \in \mathbb{R}^d.$$

COROLLARY 6.1.

$$(6.4) \quad |m_0(\vec{t})| \leq 1, \quad \forall \vec{t} \in \mathbb{R}^d.$$

(For the proof, see Appendix PROOF OF PROPOSITION 6.1.)

7. The Scaling Function φ in $L^2(\mathbb{R}^d)$

DEFINITION 7.1. *Define*

$$g(\vec{\xi}) \equiv \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^{\infty} m_0((A^j)^{-1}\vec{\xi}), \quad \forall \vec{\xi} \in \mathbb{R}^d,$$

and

$$(7.1) \quad \varphi = \mathcal{F}^{-1}g.$$

In this section we will prove that g is an $L^2(\mathbb{R}^d)$ -function. Hence by Plancherel Theorem φ is also in $L^2(\mathbb{R}^d)$. This provides the base for further construction in this paper. The proof we used in [5] (for case $d = 2$) no longer works for our general cases here. We will call φ the *scaling function*. We will use Γ_π to denote $[-\pi, \pi)^d$.

LEMMA 7.1. *Let f be a 2π -periodical continuous function on \mathbb{R}^d and let $\vec{\gamma} \in \mathbb{R}^d$. Then*

$$\int_{\Gamma_\pi + \vec{\gamma}} f(\vec{t}) d\vec{t} = \int_{\Gamma_\pi} f(\vec{t}) d\vec{t}.$$

PROOF. Denote $\Gamma_\pi^{(\vec{n})} \equiv \Gamma_\pi + \pi\vec{n}$, $\vec{n} \in \mathbb{Z}^d$. The family $\{\Gamma_\pi^{(\vec{n})}, \vec{n} \in (2\mathbb{Z})^d\}$ is a partition of \mathbb{R}^d . Since $\Gamma_\pi + \vec{\gamma}$ is bounded there is a finite subset $\vec{\Lambda} \subset (2\mathbb{Z})^d$,

$$\bigcup_{\vec{n} \in \vec{\Lambda}} (\Gamma_\pi + \vec{\gamma}) \cap \Gamma_\pi^{(\vec{n})} = \Gamma_\pi + \vec{\gamma}.$$

Denote $\Gamma_{\pi, \vec{n}} \equiv (\Gamma_\pi + \vec{\gamma}) \cap \Gamma_\pi^{(\vec{n})} - \pi\vec{n}$. We claim that the family $\{\Gamma_{\pi, \vec{n}}, \vec{n} \in \vec{\Lambda}\}$ is a partition of Γ_π . It is clear that $\Gamma_{\pi, \vec{n}} \subseteq \Gamma_\pi$, $\forall \vec{n} \in \vec{\Lambda}$. Assume that for some different $\vec{n}_1, \vec{n}_2 \in \vec{\Lambda}$, there is a vector $\vec{u} \in \Gamma_{\pi, \vec{n}_1} \cap \Gamma_{\pi, \vec{n}_2}$. This implies that $\vec{u} + \pi\vec{n}_1 \in \Gamma_\pi + \vec{\gamma}$ and $\vec{u} + \pi\vec{n}_2 \in \Gamma_\pi + \vec{\gamma}$, or $\vec{u} - \vec{\gamma} \in \Gamma_\pi^{(-\vec{n}_1)} \cap \Gamma_\pi^{(-\vec{n}_2)}$. This is impossible since $\{\Gamma_\pi^{(\vec{n})}, \vec{n} \in (2\mathbb{Z})^d\}$ is a partition of \mathbb{R}^d .

Now we have

$$\begin{aligned} \int_{\Gamma_\pi} f(\vec{t}) d\vec{t} &= \int_{\bigcup_{\vec{n} \in \vec{\Lambda}} \Gamma_{\pi, \vec{n}}} f(\vec{t}) d\vec{t} \\ &= \sum_{\vec{n} \in \vec{\Lambda}} \int_{(\Gamma_\pi + \vec{\gamma}) \cap \Gamma_\pi^{(\vec{n})} - \pi\vec{n}} f(\vec{t}) d\vec{t} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\vec{n} \in \vec{\Lambda}} \int_{(\Gamma_\pi + \vec{\gamma}) \cap \Gamma_\pi^{(\vec{n})}} f(\vec{t}) d\vec{t} \\
&= \int_{\cup_{\vec{n} \in \vec{\Lambda}} (\Gamma_\pi + \vec{\gamma}) \cap \Gamma_\pi^{(\vec{n})}} f(\vec{t}) d\vec{t} \\
&= \int_{\Gamma_\pi + \vec{\gamma}} f(\vec{t}) d\vec{t}.
\end{aligned}$$

□

LEMMA 7.2. Let M be a $d \times d$ integral matrix in \mathfrak{G} and G be a $d \times d$ integral matrix with $\det(G) \neq 0$. Then

- (1) There exists a finite partition of Γ_π , $\{\Delta_j, j \in \Lambda_M\}$ and a subset of elements of $(2\mathbb{Z})^d$, $\{\vec{n}_j, j \in \Lambda_M\}$ such that

$$(7.2) \quad M\Gamma_\pi = \bigcup_{j \in \Lambda_M} (\Delta_j + \pi \vec{n}_j).$$

- (2) Let f be a 2π -periodical continuous function on \mathbb{R}^d . Then

$$(7.3) \quad \int_{GM\Gamma_\pi} f(\vec{t}) d\vec{t} = \int_{G\Gamma_\pi} f(\vec{t}) d\vec{t}$$

and

$$(7.4) \quad \int_{M\Gamma_\pi} f(\vec{t}) d\vec{t} = \int_{\Gamma_\pi} f(\vec{t}) d\vec{t}$$

PROOF. (1) It is clear that

$$(7.5) \quad I_{ij}\Gamma_\pi = \Gamma_\pi, \quad i, j \leq d,$$

$$(7.6) \quad S_p\Gamma_\pi = \Gamma_\pi, \quad p \leq d,$$

since $I_{ij}\Gamma_\pi$, $S_p\Gamma_\pi$ have the same vertices as of Γ_π .

Let $M \equiv I + \Delta_{ij}$, $i \neq j$ and $i, j \leq d$. Since $I + \Delta_{12} = I_{2j}I_{1i}(I + \Delta_{ij})I_{1i}I_{2j}$, by (7.5) we can assume $M = I + \Delta_{12}$. The set $M\Gamma_\pi$ is a cylinder in $d \times d$ space. We have $M\Gamma_\pi = P \times [-\pi, \pi]^{d-2}$ where P is the orthogonal project of $M\Gamma_\pi$ into the two dimensional coordinate plane $X_1 \times X_2$. The set P is a parallelogram (Figure 1 left). The set P contains two disjoint triangles Δ_+ and Δ_- . P is the disjoint union of $\{\Delta_+, \Delta_-, P \setminus (\Delta_+ \cup \Delta_-)\}$. Also Γ_π is disjoint union of $\{\Delta_+ - 2\pi\vec{e}_1, \Delta_- + 2\pi\vec{e}_1, P \setminus (\Delta_+ \cup \Delta_-)\}$ (Figure 1 right).

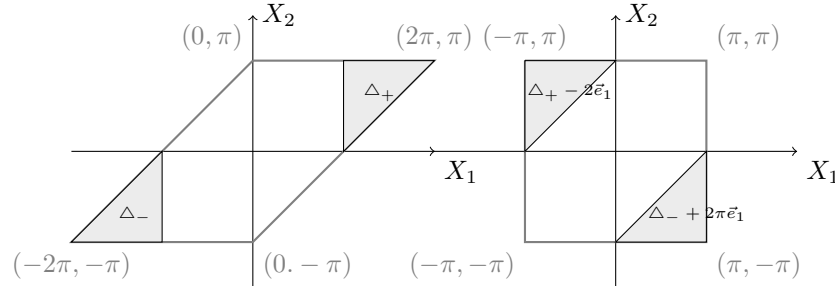


FIGURE 1. $X_1 \times X_2$ Coordinate Plane

Therefore

- $\{\Delta_+ \times [-\pi, \pi]^{d-2}, \Delta_- \times [-\pi, \pi]^{d-2}, P \setminus (\Delta_+ \cup \Delta_-) \times [-\pi, \pi]^{d-2}\}$ is a partition of $M\Gamma_\pi$;
- $\{\Delta_+ \times [-\pi, \pi]^{d-2} - 2\pi\vec{e}_1, \Delta_- \times [-\pi, \pi]^{d-2} + 2\pi\vec{e}_1, P \setminus (\Delta_+ \cup \Delta_-) \times [-\pi, \pi]^{d-2}\}$ is a partition of Γ_π .

The proof for the case $M = I - \Delta_{ij}$ is similar to this. We omit it.

(2) Since $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $\vec{t} \mapsto G\vec{t}$ is linear and one-to-one, we have

$$GM\Gamma_\pi = G\left(\bigcup (\Delta_j + \pi\vec{n}_j)\right) = \bigcup (G\Delta_j + \pi G\vec{n}_j).$$

Also, $\pi G\vec{n}_j \in \pi(2\mathbb{Z})^d$ and f is 2π -periodical, so we have

$$\begin{aligned} \int_{GM\Gamma_\pi} f(\vec{t}) d\vec{t} &= \int_{\bigcup (G\Delta_j + \pi G\vec{n}_j)} f(\vec{t}) d\vec{t} \\ &= \int_{G(\bigcup \Delta_j)} f(\vec{t}) d\vec{t} \\ &= \int_{G\Gamma_\pi} f(\vec{t}) d\vec{t} \end{aligned}$$

Replace G by I in (7.3), we obtain (7.4). □

LEMMA 7.3. *Let A be a $d \times d$ expansive integral matrix with $|\det(A)| = 2$, and let f be a 2π -periodical continuous function on \mathbb{R}^d . Then*

$$\int_{A\Gamma_\pi} f d\vec{t} = 2 \int_{\Gamma_\pi} f d\vec{t}$$

PROOF. By Proposition 3.1 Equation (3.5) we have $V_j, U_i \in \mathfrak{G}$ such that

$$A = V_1 V_2 \cdots V_{n_0} D_d U_1 U_2 \cdots U_{m_0}.$$

Repeatedly using Lemma 7.2 Equation (7.3) we have

$$\begin{aligned} \int_{A\Gamma_\pi} f d\vec{t} &= \int_{V_1 V_2 \cdots V_{n_0} D_d U_1 U_2 \cdots U_{m_0} \Gamma_\pi} f d\vec{t} \\ &= \int_{V_1 V_2 \cdots V_{n_0} D_d \Gamma_\pi} f d\vec{t}. \end{aligned}$$

Denote $\Gamma_\pi^+ \equiv [-\pi, \pi]^{d-1} \times [0, \pi]$, $\Gamma_\pi^- \equiv [-\pi, \pi]^{d-1} \times [-\pi, 0]$. Modulus a measure zero set we have $D_d \Gamma_\pi = [-\pi, \pi]^{d-1} \times [-2\pi, 2\pi] = \Gamma_\pi \cup ((\Gamma_\pi^+ - 2\pi\vec{e}_d) \cup (\Gamma_\pi^- + 2\pi\vec{e}_d))$. So we have

$$\begin{aligned} \int_{A\Gamma_\pi} f d\vec{t} &= \int_{V_1 V_2 \cdots V_{n_0} (\Gamma_\pi \cup ((\Gamma_\pi^+ - 2\pi\vec{e}_d) \cup (\Gamma_\pi^- + 2\pi\vec{e}_d)))} f d\vec{t} \\ &= \int_{V_1 V_2 \cdots V_{n_0} \Gamma_\pi} f d\vec{t} + \int_{V_1 V_2 \cdots V_{n_0} ((\Gamma_\pi^+ - 2\pi\vec{e}_d) \cup (\Gamma_\pi^- + 2\pi\vec{e}_d))} f d\vec{t}. \end{aligned}$$

By Lemma 7.2 we have

$$\int_{V_1 V_2 \cdots V_{n_0} \Gamma_\pi} f d\vec{t} = \int_{\Gamma_\pi} f d\vec{t},$$

and

$$\begin{aligned}
& \int_{V_1 V_2 \cdots V_{n_0} ((\Gamma_\pi^+ - 2\pi \vec{e}_d) \cup (\Gamma_\pi^- + 2\pi \vec{e}_d))} f d\vec{t} \\
&= \int_{V_1 V_2 \cdots V_{n_0} (\Gamma_\pi^- + 2\pi \vec{e}_d)} f d\vec{t} + \int_{V_1 V_2 \cdots V_{n_0} (\Gamma_\pi^+ - 2\pi \vec{e}_d)} f d\vec{t} \\
&= \int_{V_1 V_2 \cdots V_{n_0} (\Gamma_\pi^+ \cup \Gamma_\pi^-)} f d\vec{t} \\
&= \int_{V_1 V_2 \cdots V_{n_0} \Gamma_\pi} f d\vec{t} \\
&= \int_{\Gamma_\pi} f d\vec{t}.
\end{aligned}$$

The proof of Lemma 7.3 is complete. \square

PROPOSITION 7.1. *The functions g and φ are in $L^2(\mathbb{R}^d)$.*

PROOF. For $J \in \mathbb{N}$, we define

$$M_J(\vec{\xi}) \equiv \begin{cases} \prod_{j=1}^J |m_0((A^\tau)^{-j} \vec{\xi})|^2, & \text{if } \vec{\xi} \in (A^\tau)^{J+1} \Gamma_\pi; \\ 0, & \text{if } \vec{\xi} \in \mathbb{R}^d \setminus (A^\tau)^{J+1} \Gamma_\pi. \end{cases}$$

To prove the Proposition 7.1, by Fatou's Lemma it suffices to show that $\{\int_{\mathbb{R}^d} M_J(\vec{\xi}) d\vec{\xi}, J \in \mathbb{N}\}$ is a bounded sequence.

We have

$$\begin{aligned}
\int_{\mathbb{R}^d} M_J(\vec{\xi}) d\vec{\xi} &= \int_{(A^\tau)^{J+1} \Gamma_\pi} \prod_{k=1}^J |m_0((A^\tau)^{-k} \vec{\xi})|^2 d\vec{\xi} \\
&= \int_{(A^\tau)^J (A^\tau \Gamma_\pi)} \prod_{k=1}^J |m_0((A^\tau)^{-k} \vec{\xi})|^2 d\vec{\xi} \\
&= |\det((A^\tau)^J)| \int_{A^\tau \Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \vec{\eta})|^2 d\vec{\eta}
\end{aligned}$$

where $\vec{\eta} \equiv (A^\tau)^{-J} \vec{\xi}$.

We claim that

$$(7.7) \quad \int_{A^\tau \Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \vec{\eta})|^2 d\vec{\eta} = \int_{\Gamma_\pi} \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\eta})|^2 d\vec{\eta}.$$

By this claim and the calculation before we will have

$$\begin{aligned}
\int_{\mathbb{R}^d} M_J(\vec{\xi}) d\vec{\xi} &= |\det((A^\tau)^J)| \int_{A^\tau \Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \vec{\eta})|^2 d\vec{\eta} \\
&= |\det((A^\tau)^J)| \int_{\Gamma_\pi} \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\eta})|^2 d\vec{\eta} \\
&= \int_{(A^\tau)^J \Gamma_\pi} \prod_{m=1}^{J-1} |m_0((A^\tau)^{-(J-m)} \vec{\xi})|^2 d\vec{\xi}
\end{aligned}$$

$$= \int_{\mathbb{R}^d} M_{J-1}(\vec{\xi}) d\vec{\xi}.$$

So, $\{\int_{\mathbb{R}^d} M_J(\vec{\xi}) d\vec{\xi}, J \in \mathbb{N}\}$ is a constant sequence. We will complete the proof when we finish the proof of the claim.

Proof of the Claim. By Lemma 7.1, let $\vec{\gamma} = -\pi\vec{q}_A$, we have

$$\begin{aligned} & \int_{\Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} \\ &= \int_{\Gamma_\pi - \pi\vec{q}_A} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} \\ &= \int_{\Gamma_\pi} |m_0(\vec{\lambda} + \pi\vec{q}_A)|^2 \cdot \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\lambda} + \pi(A^\tau)^m \vec{q}_A)|^2 d\vec{\lambda} \end{aligned}$$

where $\vec{\lambda} = \vec{\eta} - \pi\vec{q}_A$.

By Partition Theorem equation (4.3), $\vec{q}_A \circ A\mathbb{Z}^d$ are even numbers. So $A^\tau \vec{q}_A \circ \vec{n}$ is even for every $\vec{n} \in \mathbb{Z}^d$. This implies that $A^\tau \vec{q}_A \in (2\mathbb{Z})^d$. So $(A^\tau)^m \vec{q}_A \in (2\mathbb{Z})^d$ for $m \geq 1$. By the fact that the function m_0 is 2π -periodical, we have

$$(7.8) \quad \int_{\Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} = \int_{\Gamma_\pi} |m_0(\vec{\lambda} + \pi\vec{q}_A)|^2 \cdot \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\lambda})|^2 d\vec{\lambda}$$

Now, by Lemma 7.3 we have

$$(7.9) \quad \int_{A^\tau \Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} = 2 \cdot \int_{\Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta}.$$

Combine (7.9) and (7.8), and by (6.2) we have

$$\begin{aligned} & \int_{A^\tau \Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} \\ &= \int_{\Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} + \int_{\Gamma_\pi} \prod_{m=0}^{J-1} |m_0((A^\tau)^m \eta)|^2 d\vec{\eta} \\ &= \int_{\Gamma_\pi} |m_0(\vec{\lambda})|^2 \cdot \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\lambda})|^2 d\vec{\lambda} + \int_{\Gamma_\pi} |m_0(\vec{\lambda} + \pi\vec{q}_A)|^2 \cdot \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\lambda})|^2 d\vec{\lambda} \\ &= \int_{\Gamma_\pi} (|m_0(\vec{\lambda})|^2 + |m_0(\vec{\lambda} + \pi\vec{q}_A)|^2) \cdot \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\lambda})|^2 d\vec{\lambda} \\ &= \int_{\Gamma_\pi} \prod_{m=1}^{J-1} |m_0((A^\tau)^m \vec{\lambda})|^2 d\vec{\lambda}. \end{aligned}$$

The claim has been proven. □

8. φ has a compact support

In this section we will prove that the scaling function φ has a compact support in \mathbb{R}^d (Proposition 8.2). We outline the ideas for this. We place the proofs of Lemma 8.1, Lemma 8.3 and Proposition 8.2 in the Appendix since in the proofs we use the similar ideas we used in the previous paper [5]

We will need the following Schwartz's Paley-Wiener Theorem.

SCHWARTZ'S PALEY-WIENER THEOREM *An entire function F on \mathbb{C}^d , $d \in \mathbb{N}$, is the Fourier Transform of a distribution with compact support in \mathbb{R}^d if and only if there are some constants C, N and B , such that*

$$(8.1) \quad |F(\vec{\xi})| \leq C(1 + |\vec{\xi}|)^N e^{B|\Im(\vec{\xi})|}, \quad \forall \vec{\xi} \in \mathbb{C}^d$$

The distribution is supported on the closed ball with center $\vec{0}$ and radius B .

First, we prove that g is an entire function. Denote $d_j(\vec{\xi}) \equiv m_0((A^\tau)^{-j}\vec{\xi}) - 1$ and denote $\beta = \|(A^\tau)^{-1}\|^{-1}$. Since A is expansive, $\beta > 1$. Here $\|\cdot\|$ is the operator norm of linear operators on the Euclidian space \mathbb{C}^d .

LEMMA 8.1. *Let Ω be a bounded closed region in \mathbb{C}^d . Then there exists a constant $C_\Omega > 0$,*

$$(8.2) \quad |d_j(\vec{\xi})| \leq \frac{C_\Omega}{\beta^j}, \quad \forall j \in \mathbb{N}, \vec{\xi} \in \Omega.$$

(For the proof, see Appendix PROOF OF LEMMA 8.1.)

PROPOSITION 8.1. *The function $g(\vec{\xi})$ is an entire function on \mathbb{C}^d .*

PROOF. For $J \in \mathbb{N}$, $\vec{\xi} \in \mathbb{C}^d$, define

$$(8.3) \quad g_J(\vec{\xi}) \equiv \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^J m_0((A^\tau)^{-j}\vec{\xi}) = \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^J (1 + d_j(\vec{\xi})).$$

Since this is a finite product, g_J is an entire function.

Since $\beta > 1$, by Lemma 8.1 $\sum |d_j(\vec{\xi})|$ converges uniformly on bounded region Ω , the product $\prod_{j=0}^\infty (1 + |d_j(\vec{\xi})|)$ converges uniformly on Ω . This implies that g is the uniform limit of a sequence of entire functions g_J on every bounded region Ω . By Morera Theorem g is an entire function on \mathbb{C}^d . □

LEMMA 8.2.

$$(8.4) \quad |e^{-iz} - 1| \leq \min(2, |z|), \quad \forall z \in \mathbb{C}, \Im(z) \leq 0.$$

PROOF. Let $z = a + ib$, with $b = \Im(z) \leq 0$. So we have

$$|e^{-iz} - 1| \leq 1 + |e^{-iz}| \leq 1 + e^b \leq 2.$$

On the other hand, we have $e^b > 1 + b$, $\forall b \neq 0$. When $b < 0$, $b^2 = (-b)^2 > (1 - e^b)^2$. So

$$|e^{-iz} - 1|^2 = e^{2b} - 2e^b \cos a + 1 = (e^b - 1)^2 + e^b \cdot 4 \sin^2 \frac{a}{2} \leq b^2 + a^2 = |z|^2.$$

□

LEMMA 8.3. *There exist constants B_0, C_0 such that for all $j \in \mathbb{N}$, $\vec{\xi} \in \mathbb{C}^d$,*

$$(8.5) \quad |m_0((A^\tau)^{-j}\vec{\xi})| \leq (1 + C_0 \min(1, \frac{|\vec{\xi}|}{\beta^j})) \exp(\frac{B_0|\Im(\vec{\xi})|}{\beta^j}).$$

(For the proof, see Appendix PROOF OF LEMMA 8.3.)

PROPOSITION 8.2. *The scaling function φ is an $L^2(\mathbb{R}^d)$ function with compact support.*

(For the proof, see Appendix PROOF OF PROPOSITION 8.2.)

9. Parseval's Frame Wavelet Function ψ

In this section we will define a function ψ associated with the scaling function φ . In Theorem 9.1 we prove that the function ψ is a Parseval's frame wavelet function associated with matrix A . Since in the proofs of Lemma 9.2, Proposition 9.2 and Proposition 9.3 we use the ideas that we have used in the previous paper [5], to make this paper readable, we place these proofs in Appendix.

By Definition 7.1 and Equations (5.3), (5.4) and (5.6) in Lemma 5.1 we have

$$\begin{aligned} \widehat{\varphi}(\vec{s}) &= g(\vec{s}) = m_0((A^\tau)^{-1}\vec{s}) \cdot \frac{1}{(2\pi)^{d/2}} \prod_{j=2}^{\infty} m_0((A^\tau)^{-j}\vec{s}) \\ &= m_0((A^\tau)^{-1}\vec{s}) g((A^\tau)^{-1}\vec{s}) \\ &= \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} e^{-i\vec{n} \circ (A^\tau)^{-1}\vec{s}} g((A^\tau)^{-1}\vec{s}) \\ &= \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} \widehat{T}_{A^{-1}\vec{n}} \widehat{D}_A g(\vec{s}), \text{ by (5.4) and (5.7)} \\ &= \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} \widehat{D}_A \widehat{T}_{\vec{n}} \widehat{\varphi}(\vec{s}), \text{ by (5.3).} \end{aligned}$$

Taking Fourier inverse transform on two sides, we have

$$(9.1) \quad \varphi = \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_A T_{\vec{n}} \varphi.$$

This is the *two scaling equation associate with matrix A* . An equivalent form is

$$\varphi(\vec{t}) = \sqrt{2} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} \varphi(A\vec{t} - \vec{n}), \quad \vec{t} \in \mathbb{R}^d.$$

DEFINITION 9.1. *Define a function ψ on \mathbb{R}^d by*

$$(9.2) \quad \psi \equiv \sum_{\vec{n} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} D_A T_{\vec{n}} \varphi.$$

An equivalent statement is

$$\psi(\vec{t}) = \sqrt{2} \sum_{\vec{n} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} \varphi(A\vec{t} - \vec{n}), \quad \forall \vec{t} \in \mathbb{R}^d.$$

It is clear that this function ψ has a compact support since φ has a compact support and $\{h_{\vec{n}}\}$ is a finite solution to Lawton's equations. For $J \in \mathbb{Z}$, and $f \in L^2(\mathbb{R}^d)$ define

$$\begin{aligned}
I_J &\equiv \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^J T_{\vec{k}} \varphi \rangle D_A^J T_{\vec{k}} \varphi; \\
F_J &\equiv \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^J T_{\vec{k}} \psi \rangle D_A^J T_{\vec{k}} \psi.
\end{aligned}$$

PROPOSITION 9.1. *Let $f \in L^2(\mathbb{R}^d)$. Then*

$$(9.3) \quad I_{J+1} = I_J + F_J, \forall J \in \mathbb{Z}.$$

(For the proof, see Appendix PROOF OF PROPOSITION 9.1.)

For $f \in L^2(\mathbb{R}^d)$ and $J \in \mathbb{Z}$, we will use the following notations.

$$\begin{aligned}
L_0(f) &\equiv \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, T_{\vec{\ell}} \varphi \rangle|^2. \\
L_J(f) &\equiv \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^J T_{\vec{\ell}} \varphi \rangle|^2 = L_0((D_A^J)^* f).
\end{aligned}$$

Let $\rho > 0$, we define functions f_ρ and $f_{\bar{\rho}}$ by

$$\begin{aligned}
\widehat{f}_\rho &\equiv \widehat{f} \cdot \chi_{\{|\vec{v}| \leq \rho\}}, \\
\widehat{f}_{\bar{\rho}} &\equiv \widehat{f} \cdot \chi_{\{|\vec{v}| > \rho\}}.
\end{aligned}$$

Here χ is the characteristic function. It is clear that we have

LEMMA 9.1. *The function f_ρ and $f_{\bar{\rho}}$ have the following properties,*

- (1) $f = f_\rho + f_{\bar{\rho}}$,
- (2) $\|f\|^2 = \|\widehat{f}\|^2 = \|f_\rho\|^2 + \|f_{\bar{\rho}}\|^2$,
- (3) $\lim_{\rho \rightarrow \infty} \|f_\rho\|^2 = \|f\|^2$,
- (4) $\lim_{\rho \rightarrow \infty} \|f_{\bar{\rho}}\|^2 = 0$.

THEOREM 9.1. *Let ψ be as defined in Definition 9.1. Then, $\{D_A^n T_{\vec{\ell}} \psi, n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d\}$ is a Parseval's frame for $L^2(\mathbb{R}^d)$.*

PROOF. Let $f \in L^2(\mathbb{R}^d)$. We will prove that

$$(9.4) \quad f = \sum_{n \in \mathbb{Z}} \sum_{\vec{\ell} \in \mathbb{Z}^d} \langle f, D_A^n T_{\vec{\ell}} \psi \rangle D_A^n T_{\vec{\ell}} \psi,$$

the convergence is in $L^2(\mathbb{R}^d)$ -norm.

By Proposition 9.1, we have $I_j - I_{j-1} = F_{j-1}$, $\forall j \in \mathbb{Z}$. Hence

$$\sum_{j=-J+1}^J F_j = I_J - I_{-J}, \forall J \in \mathbb{Z}.$$

This implies that

$$\begin{aligned}
&\sum_{j=-J+1}^J \sum_{\vec{\ell} \in \mathbb{Z}^d} \langle f, D_A^j T_{\vec{\ell}} \psi \rangle D_A^j T_{\vec{\ell}} \psi \\
&= \sum_{\vec{\ell} \in \mathbb{Z}^d} \langle f, D_A^J T_{\vec{\ell}} \varphi \rangle D_A^J T_{\vec{\ell}} \varphi - \sum_{\vec{\ell} \in \mathbb{Z}^d} \langle f, D_A^{-J} T_{\vec{\ell}} \varphi \rangle D_A^{-J} T_{\vec{\ell}} \varphi.
\end{aligned}$$

Taking inner product of f with both sides of the equation, we have

$$\sum_{j=-J+1}^J \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^j T_{\vec{\ell}} \psi \rangle|^2 = L_J(f) - L_{-J}(f).$$

By Proposition 9.2 and Proposition 9.3, we have

$$\begin{aligned} \lim_{J \rightarrow +\infty} L_J(f) &= \|f\|^2; \\ \lim_{J \rightarrow +\infty} L_{-J}(f) &= 0. \end{aligned}$$

So, we have

$$\sum_{j \in \mathbb{Z}} \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^j T_{\vec{\ell}} \psi \rangle|^2 = \|f\|^2, \forall f \in L^2(\mathbb{R}^d).$$

□

PROPOSITION 9.2. *Let $f \in L^2(\mathbb{R}^d)$. Then*

$$\lim_{J \rightarrow +\infty} L_{-J}(f) = 0.$$

(For the proof, see Appendix PROOF OF PROPOSITION 9.2.)

LEMMA 9.2. *Let $f \in L^2(\mathbb{R}^d)$ and $J \in \mathbb{Z}$. Then*

$$L_J(f) = (2\pi)^d \int_{\mathbb{R}^d} \sum_{\vec{\ell} \in \mathbb{Z}^d} \left(\widehat{f}(\vec{t}) \overline{\widehat{f}(\vec{t} - 2\pi(A^\tau)^J \vec{\ell})} \widehat{\varphi}((A^\tau)^{-J} \vec{t} - 2\pi \vec{\ell}) \overline{\widehat{\varphi}((A^\tau)^{-J} \vec{t})} \right) d\vec{t}$$

(For the proof, see Appendix PROOF OF LEMMA 9.2.)

PROPOSITION 9.3. *We have*

$$(9.5) \quad \lim_{J \rightarrow +\infty} L_J(f) = \|f\|^2, \forall f \in L^2(\mathbb{R}^d).$$

(For the proof, see Appendix PROOF OF PROPOSITION 9.3.)

10. Summary and Iteration

Let B be a $d \times d$ expansive integral matrix with $\det B = \pm 2$ and f be a function in $L^2(\mathbb{R}^d)$. Denote

$$\begin{aligned} V_B^{(0)}(f) &\equiv \overline{\text{span}}(\{T_{\vec{\ell}} f, \vec{\ell} \in \mathbb{Z}^d\}). \\ V_B^{(n)}(f) &\equiv D_B^n V_B^{(0)}(f), n \in \mathbb{Z}. \end{aligned}$$

Let A_0 be an $d \times d$ integral expansive matrix with $\det A_0 = \pm 2$. By the Partition Theorem (Theorem 4.1) $A_0 = S^{-1}AS$ for some $d \times d$ integral matrix S with $\det S = \pm 1$ and there are two vectors $\vec{\ell}_A, \vec{q}_A \in \mathbb{Z}^d$ with properties (1)-(5) in Theorem 4.1. By the construction in Sections §4-§9 we obtained scaling function φ_A and Parseval's frame wavelet in ψ_A $L^2(\mathbb{R}^d)$.

Define φ_{A_0} and ψ_{A_0} by $\varphi_{A_0}(\vec{t}) \equiv (U_S \varphi_A)(\vec{t}) = \varphi_A(S\vec{t})$ and $\psi_{A_0}(\vec{t}) \equiv (U_S \psi_A)(\vec{t}) = \psi_A(S\vec{t})$. We have

THEOREM 10.1. (1) *The function ψ_{A_0} is a Parseval's frame wavelet associated with matrix A_0 .*

(2)

$$(10.1) \quad \varphi_{A_0} = \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_{A_0} T_{S^{-1}\vec{n}} \varphi_{A_0}.$$

$$(10.2) \quad \psi_{A_0} = \sum_{\vec{\ell}_A - \vec{n} \in \Lambda_0} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} D_{A_0} T_{S^{-1}\vec{n}} \varphi_{A_0}.$$

where $\{h_{\vec{n}}, \vec{n} \in \Lambda_0\}$ is the finite solution to the Lawton's system of equations associated with matrix A ; vectors $\{\ell_A, \vec{q}_A\}$ and the $d \times d$ matrix S are as defined in the Partition Theorem (Theorem 4.1).

(3)

$$V_{A_0}^{(n)}(\varphi_{A_0}) \subset V_{A_0}^{(n+1)}(\varphi_{A_0}), \quad n \in \mathbb{Z}.$$

(4)

$$\overline{\bigcup_{n \in \mathbb{Z}} V_{A_0}^{(n)}(\varphi_{A_0})} = L^2(\mathbb{R}^d).$$

PROOF. By Equations (9.1) and (9.2) we have

$$\begin{aligned} \varphi_A &= \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} D_A T_{\vec{n}} \varphi_A. \\ \psi_A &= \sum_{\vec{n} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} D_A T_{\vec{n}} \varphi. \end{aligned}$$

This implies that for any $\vec{k} \in \mathbb{Z}^d$,

$$T_{\vec{k}} \varphi_A = T_{\vec{k}} D_A \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} T_{\vec{n}} \varphi_A = D_A \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} T_{\vec{n} + A\vec{k}} \varphi_A \in V_A^{(1)}(\varphi_A)$$

So we have

$$V_A^{(0)}(\varphi_A) \subset V_A^{(1)}(\varphi_A) \text{ and } \psi_A \in V_A^{(1)}(\varphi_A).$$

Hence

$$V_A^{(n)}(\varphi_A) \subset V_A^{(n+1)}(\varphi_A), \forall n \in \mathbb{Z}, \text{ and } \{T_{\vec{k}} \psi_A, \vec{k} \in \mathbb{Z}^d\} \subset V_A^{(1)}(\varphi_A).$$

Therefore, we have

$$\{D_A^m T_{\vec{k}} \psi_A, \vec{k} \in \mathbb{Z}^d\} \subset V_A^{(1)}(\varphi_A), \forall m \in \mathbb{Z}, m \leq 0.$$

Apply D_A^n to the two sides and take union, we obtain,

$$\{D_A^n T_{\vec{k}} \psi_A, n \in \mathbb{Z}, \vec{k} \in \mathbb{Z}^d\} \subset \bigcup_{n \in \mathbb{Z}} V_A^{(n)}(\varphi_A).$$

This equation together with Theorem 9.1 Equation (9.4) we get $\overline{\bigcup_{n \in \mathbb{Z}} V_A^{(n)}(\varphi_A)} = L^2(\mathbb{R}^d)$.

Next, notice that Λ_0 is the finite support of the solution $\{h_{\vec{n}}\}$ to the Lawton's system of equations (6.1) associated with matrix A , by Equations (9.1), (9.2), (5.1) and (5.2), we have

$$\varphi_{A_0} = U_S \varphi_A = U_S \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} D_A T_{\vec{n}} \varphi_A$$

$$\begin{aligned}
&= \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} (U_S D_A U_S^{-1}) (U_S T_{\vec{n}} U_S^{-1}) (U_S \varphi_A) \\
&= \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} D_{A_0} T_{S^{-1}\vec{n}} \varphi_{A_0} \\
&= \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_{A_0} T_{S^{-1}\vec{n}} \varphi_{A_0}. \\
\psi_{A_0} &= U_S \psi_A = U_S \sum_{\vec{n} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} D_A T_{\vec{n}} \varphi \\
&= \sum_{\vec{n} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} (U_S D_A U_S^{-1}) (U_S T_{\vec{n}} U_S^{-1}) (U_S \varphi_A) \\
&= \sum_{\vec{\ell}_A - \vec{n} \in \Lambda_0} (-1)^{\vec{q}_A \circ \vec{n}} \overline{h_{\vec{\ell}_A - \vec{n}}} D_{A_0} T_{S^{-1}\vec{n}} \varphi_{A_0}.
\end{aligned}$$

Since U_S is a unitary operator. It will map a normalize tight frame into a normalized tight frame. We have $U_S \{D_A^n T_{\vec{k}} \psi_A, n \in \mathbb{Z}, \vec{k} \in \mathbb{Z}^d\} = \{D_{A_0}^n T_{\vec{k}} \psi_{A_0}, n \in \mathbb{Z}, \vec{k} \in \mathbb{Z}^d\}$. So ψ_{A_0} is a Parseval's frame wavelet. It is also true that $U_S V_A^{(n)}(\varphi_A) = V_{A_0}^{(n)}(\varphi_{A_0}), \forall n \in \mathbb{Z}$. So we have (3) and (4). \square

Let $f_0(\vec{t})$ be a bounded function in $L^2(\mathbb{R}^d)$ which is contiguous at $\vec{0}$ and $f_0(\vec{0}) = 1$. Define

$$\begin{aligned}
g_0(\vec{\xi}) &\equiv \frac{1}{(2\pi)^{d/2}} \cdot f_0(\vec{\xi}), \\
g_k(\vec{\xi}) &\equiv \frac{1}{(2\pi)^{d/2}} \cdot f_0((A^\tau)^{-k} \vec{\xi}) \cdot \prod_{j=1}^k m_0((A^\tau)^{-j} \vec{\xi}), \quad \forall k \geq 1,
\end{aligned}$$

and

$$\varphi_k \equiv \mathcal{F}^{-1} g_k, \forall k \geq 0.$$

Since $\lim_k f_0((A^\tau)^{-k} \vec{\xi})$ is converging to the constant function 1 uniformly on any given bounded region of \mathbb{R}^d and $\prod_{j=1}^k m_0((A^\tau)^{-j} \vec{\xi})$ is also converging uniformly on any given bounded region of $L^2(\mathbb{R}^d)$ (see the proof of Proposition 8.1), we have $\lim g_k = g$ and $\lim \varphi_k = \varphi$. The convergence is in $L^2(\mathbb{R}^d)$ -norm. We have

$$\begin{aligned}
\widehat{\varphi}_{k+1}(\vec{s}) &= g_{k+1}(\vec{s}) \\
&= m_0((A^\tau)^{-1} \vec{s}) \cdot \frac{1}{(2\pi)^{d/2}} \cdot f_0((A^\tau)^{-(k+1)} \vec{s}) \cdot \prod_{j=2}^{k+1} m_0((A^\tau)^{-j} \vec{s}) \\
&= m_0((A^\tau)^{-1} \vec{s}) \cdot g_k((A^\tau)^{-1} \vec{s}) \\
&= \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} e^{-i\vec{n} \circ (A^\tau)^{-1} \vec{s}} g_k((A^\tau)^{-1} \vec{s}) \\
&= \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} \widehat{T}_{A^{-1}\vec{n}} \widehat{D}_A g_k(\vec{s}), \text{ by (5.4) and (5.7)} \\
&= \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} \widehat{D}_A \widehat{T}_{\vec{n}} \widehat{\varphi}_k(\vec{s}), \text{ by (5.3)}.
\end{aligned}$$

Hence, we have

$$(10.3) \quad \varphi_{k+1} = \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_A T_{\vec{n}} \varphi_k, \quad k = 1, 2, \dots,$$

and

$$(10.4) \quad \varphi_1 = \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_A T_{\vec{n}} \varphi_0.$$

Let Ψ be the linear operator

$$(10.5) \quad \Psi \equiv \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} D_A T_{\vec{n}}.$$

The above discussion proves that if $\widehat{\varphi}_0$ is a bounded $L^2(\mathbb{R}^d)$ -function which is continuous at $\vec{0}$ with value 1, then the sequence $\{\Psi^k \varphi_0\}$ converges to the scaling function φ in $L^2(\mathbb{R}^d)$ -norm.

THEOREM 10.2. *We have*

$$(10.6) \quad \lim_{k \rightarrow \infty} \Psi^k \chi_{[0,1]^d} = \varphi$$

the limit converges in $L^2(\mathbb{R}^d)$ -norm.

PROOF. Let $\varphi_0 \equiv \chi_{[0,1]^d}$. All we need to prove is that $\mathcal{F}\varphi_0 = \frac{1}{(2\pi)^{d/2}} \cdot f_0$ for some function f_0 and this f_0 is bounded in \mathbb{R}^d and continuous at $\vec{0}$. We have

$$\begin{aligned} \mathcal{F}\varphi_0(\vec{s}) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\vec{s} \circ \vec{t}} \chi_{[0,1]^d} d\vec{t} \\ &= \frac{1}{(2\pi)^{d/2}} \int_{[0,1]^d} e^{-i\vec{s} \circ \vec{t}} d\vec{t} \\ &= \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^d \left(\frac{e^{is_j} - 1}{s_j} \right) \end{aligned}$$

The function $\frac{e^{is_j} - 1}{s_j}$ is continuous at $s_j = 0$ if we define its value by 1. Also, it is bounded when s_j is real numbers. This proves this theorem. \square

11. Examples

In this section we will use our methods developed in this paper to construction some examples of wavelet functions in $L^2(\mathbb{R}^3)$.

EXAMPLE 1. In this example we will construct a Haar wavelet in $L^2(\mathbb{R}^3)$ associated with matrix

$$A_0 \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

We have $A_0 = S^{-1}AS$ where

$$A \equiv \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S \equiv \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\det(A) = 2$ with eigenvalues $\{\sqrt[3]{2}e^{\frac{ik\pi}{3}}, k = 0, 1, 2\}$. The matrix A is expansive. We have

$$A\mathbb{Z}^3 = \left\{ \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (2\mathbb{Z})^3, \alpha, \beta \in \mathbb{Z} \right\} = A^\tau \mathbb{Z}^3.$$

Let

$$\vec{\ell}_A \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{q}_A \equiv \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The vectors $\vec{\ell}_A, \vec{q}_A$ and matrix A have the properties (1)-(5) in the Partition Theorem. In this example we assume that the only non zero elements for $h_{\vec{n}}$ are at

$$\vec{n}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \vec{\ell}_A + A\mathbb{Z}^3.$$

So the product $h_{\vec{n}_0} \bar{h}_{\vec{n}_1}$ is not in any of the Equations (6.1). The reduced system of Lawton's Equations is

$$\begin{cases} h_{\vec{n}_0}^2 + h_{\vec{n}_1}^2 = 1 \\ h_{\vec{n}_0} + h_{\vec{n}_1} = \sqrt{2}. \end{cases}$$

The system has one solution $h_{\vec{n}_0} = h_{\vec{n}_1} = \frac{\sqrt{2}}{2}$. The two scaling relation equation (9.1) is

$$(11.1) \quad \varphi_A = \frac{\sqrt{2}}{2} D_A (I + T_{\vec{n}_1}) \varphi_A.$$

By Theorem 10.1

$$\varphi_{A_0} = U_S \varphi_A = \frac{\sqrt{2}}{2} D_{A_0} (I + T_{S^{-1}\vec{n}_1}) \varphi_{A_0} = \frac{\sqrt{2}}{2} D_{A_0} (I + T_{\vec{e}_3}) \varphi_{A_0}.$$

Notice that we have $(I + T_{\vec{e}_3})\chi_{[0,1]^3} = \chi_{[0,1]^2 \times [0,2]}$ and $\frac{\sqrt{2}}{2} D_{A_0} \chi_{[0,1]^2 \times [0,2]} = \chi_{[0,1]^3}$. The function $\chi_{[0,1]^3}$ is the scaling function φ_{A_0} . Then the related normalized tight frame (orthogonal) wavelet is

$$\psi_{A_0} = \chi_{Q^+} - \chi_{Q^-}, \text{ with } Q^+ \equiv \chi_{[0,0.5] \times [0,1]^2} \text{ and } Q^- \equiv \chi_{[0.5,1] \times [0,1]^2}$$

This is a Haar wavelet in $L^2(\mathbb{R}^3)$. By this method, we can find examples of Haar wavelets in any dimension.

EXAMPLE 2. Let

$$A \equiv \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad \vec{\ell}_A \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{q}_A \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is clear that $\det(A) = -2$. Also, we have

$$A\mathbb{Z}^3 = \{ \alpha \vec{e}_1 + \beta \vec{e}_2 + (2\mathbb{Z})^3, \alpha, \beta \in \mathbb{Z} \} = A^\tau \mathbb{Z}^3.$$

The vectors $\vec{\ell}_A, \vec{q}_A$ and matrix A satisfy the properties (1)-(5) in the Partition Theorem (Theorem 4.1). So, if we have a finite solution to the the Equations (6.1), we will have a Parseval's frame wavelet associated with matrix A . In this example

we assume $\Lambda_0 \equiv \{\vec{n} = \alpha\vec{e}_1 + \beta\vec{e}_2 + \gamma\vec{e}_3, \alpha = 0, 1, 2, 3, \beta = 0, 1, \gamma = 0, 1, \}$. The corresponding reduced Lawton's system of equations is

$$\left\{ \begin{array}{ll} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}}^2 & = 1, \\ \sum_{k=0}^3 (h_{k,0,0} \cdot h_{(1+k),0,0} + h_{k,0,1} \cdot h_{(1+k),0,1} + h_{k,1,0} \cdot h_{(1+k),1,0} + h_{k,1,1} \cdot h_{(1+k),1,1}) & = 0, \\ \sum_{k=0}^2 (h_{k,0,0} \cdot h_{(2+k),0,0} + h_{k,0,1} \cdot h_{(2+k),0,1} + h_{k,1,0} \cdot h_{(2+k),1,0} + h_{k,1,1} \cdot h_{(2+k),1,1}) & = 0, \\ \sum_{k=0}^1 (h_{k,0,0} \cdot h_{(3+k),0,0} + h_{k,0,1} \cdot h_{(3+k),0,1} + h_{k,1,0} \cdot h_{(3+k),1,0} + h_{k,1,1} \cdot h_{(3+k),1,1}) & = 0, \\ \sum_{k=0}^0 (h_{k,0,0} \cdot h_{(3+k),1,0} + h_{k,0,1} \cdot h_{(3+k),1,1}) & = 0, \\ \sum_{k=0}^1 (h_{k,0,0} \cdot h_{(2+k),1,0} + h_{k,0,1} \cdot h_{(2+k),1,1}) & = 0, \\ \sum_{k=0}^2 (h_{k,0,0} \cdot h_{(1+k),1,0} + h_{k,0,1} \cdot h_{(1+k),1,1}) & = 0, \\ \sum_{k=0}^3 (h_{k,0,0} \cdot h_{k,1,0} + h_{k,0,1} \cdot h_{k,1,1}) & = 0, \\ \sum_{k=1}^3 (h_{k,0,0} \cdot h_{(k-1),1,0} + h_{k,0,1} \cdot h_{(k-1),1,1}) & = 0, \\ \sum_{k=2}^3 (h_{k,0,0} \cdot h_{(k-2),1,0} + h_{k,0,1} \cdot h_{(k-2),1,1}) & = 0, \\ \sum_{k=3}^3 (h_{k,0,0} \cdot h_{(k-3),1,0} + h_{k,0,1} \cdot h_{(k-3),1,1}) & = 0, \\ \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} & = \sqrt{2}. \end{array} \right.$$

In the Table 1 we have two sets of solutions. The solutions satisfies the equations with errors less than 10^{-10} .

α	β	γ	Solution set 1	Solution set 2
0	0	0	0.00000000000000003754	-0.0000000000000000294
1	0	0	0.08378339374280850000	0.03292120287539430000
2	0	0	0.49453510790101500000	-0.13290357845020300000
3	0	0	0.000000000000000024969	0.000000000000000017890
0	1	0	0.00000000000000002218	0.00000000000000004947
1	1	0	0.35330635188230000000	0.55716952051625900000
2	1	0	-0.22451807131547000000	0.24991965790058100000
3	1	0	0.000000000000000011746	-0.00000000000000000691
0	0	1	0.00000000000000007270	-0.00000000000000000396
1	0	1	0.16226597620431900000	0.04430091724524290000
2	0	1	-0.25534514772672400000	0.09876422297993930000
3	0	1	-0.000000000000000012892	-0.000000000000000013295
0	1	1	0.00000000000000004295	0.00000000000000006657
1	1	1	0.68425970262500800000	0.74976363753740400000
2	1	1	0.11592624905984000000	-0.18572201823152200000
3	1	1	-0.00000000000000006065	0.00000000000000000514

TABLE 1. Two solutions to Equations

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12. Appendix

PROOF OF LEMMA 3.2

PROOF. (1) Right multiply some S_i to B to make the elements in the first row of the product all non negative. We denote the product by B_1 , and the product of the S_i we used by U_1 . U_1 is in \mathfrak{M} . Note that $S_i^{-1} = S_i$. We have

$$B = B_1 \cdot U_1.$$

(2) Let b_{1j} be the elements of the first row of B_1 . Since B is non singular, the row has some positive terms. Let b_{1j_1} be the smallest positive element in the row. So we can further factor B as following

$$B = B_2 \cdot U_2$$

where $B_2 = B_1 I_{1j_1}$ and $U_2 = I_{1j_1} U_1$. Let c_{1j} be the first row of B_2 . Now c_{11} is the smallest positive element in the row. U_2 is in \mathfrak{M} . If for some $j_2 > 1$, $c_{1j_2} > 0$, we factor further

$$B = B_3 \cdot U_3$$

where $B_3 = B_2(1 - \Delta_{1j_2})$ and $U_3 = (1 + \Delta_{1j_2})U_2$. If $c_{1j} = 0, j \geq 2$, we are done for this step.

(3) If needed, repeat above process (2), until we get factor $B = L_1 V_1$, where $L_1 = (\ell_{ij})$ is an integral matrix with $\ell_{11} > 0$ and $\ell_{1,j} = 0, j > 1$. And V_1 is a finite product of elements from \mathfrak{G} and $V_1 \in \mathfrak{M}$.

(4) We can continue the above steps on the $(d-1) \times (d-1)$ matrix $(\ell_{ij})_{i,j \geq 2}$. By induction we will have

$$B = LV,$$

where $V \in \mathfrak{M}$ and L an integral lower triangular matrix with positive diagonals $\{\ell_{11}, \dots, \ell_{dd}\}$. \square

PROOF OF LEMMA 3.3

PROOF. Let B be an integral matrix with $\det B = \pm 1$. By Lemma 3.2, we have $B = LV$ where $L = (\ell_{ij})$ is lower triangular. So $\ell_{ij} = 0$, if $i < j$ and $\ell_{ii} > 0, i = 1, \dots, d$, and $\det(L) = \prod \ell_{jj} = 1$. Therefore, every diagonal element has value 1.

Let $i, j \leq d, i > j$. Assume $\ell_{ij} > 0$. By Equation (3.1), $\Delta_{ij}^n = 0, \forall n \geq 2$. We have $(I - \Delta_{ij})^{\ell_{ij}} = I + \sum_{m=1}^{\ell_{ij}} (-1)^m C_{\ell_{ij}}^m (\Delta_{ij})^m = (I - \ell_{ij} \Delta_{ij})$. So $(I - \Delta_{ij})^{\ell_{ij}} = (I - \ell_{ij} \Delta_{ij})$. By Equation (3.2), this equation is also valid for case ℓ_{ij} . Hence we have

$$(12.1) \quad (I \pm \Delta_{ij})^{\ell_{ij}} = (I \pm \ell_{ij} \Delta_{ij}).$$

The inverse to $(I \pm \Delta_{ij})^{\ell_{ij}}$ are $(I \mp \ell_{ij} \Delta_{ij})$.

Right multiply $(I - \ell_{dj} \Delta_{dj})$ to L will subtract the d^{th} column from the j^{th} column in matrix L . Since $\ell_{dd} = 1$, we have factorization This will result a 0 value at dj position of the product and all other entries are remain unchanged. So,

$$\begin{aligned} B &= LV = (L \Pi_{j < d} (I - \Delta_{dj})^{\ell_{dj}}) (\Pi_{j < d} (I + \Delta_{dj})^{\ell_{dj}} V) \\ &= (L \Pi_{j < d} (I - \Delta_{dj})^{\ell_{dj}} \cdot \Pi_{j < d-1} (I - \Delta_{(d-1)j})^{\ell_{(d-1)j}} \dots \Pi_{j < 2} (I - \Delta_{2j})^{\ell_{2j}}) \cdot \\ &\quad (\Pi_{i=2}^d \Pi_{j < i} (I + \Delta_{ij})^{\ell_{ij}} V) \end{aligned}$$

The product in the first big parentheses is I since every $\ell_{ij}, i > j$ is changed into 0. By Lemma 3.2, V is the finite product of elements from \mathfrak{G} , hence, B can be factored as products of elements from \mathfrak{G} . \square

PROOF OF THEOREM 5.1.

PROOF. We have $B = S^{-1}AS, D_B = U_B = U_{S^{-1}AS} = U_S U_A U_{S^{-1}} = U_S D_A U_S^{-1}$. Let $f \in L^2(\mathbb{R}^d)$. Since ψ_A is a Parseval's frame wavelet associated with the matrix A , we have

$$U_S^{-1} f = \sum_{n \in \mathbb{Z}, \tilde{\ell} \in \mathbb{Z}^d} \langle U_S^{-1} f, D_A^n T_{\tilde{\ell}} \psi_A \rangle D_A^n T_{\tilde{\ell}} \psi_A.$$

Then

$$\begin{aligned} f &= \sum_{n \in \mathbb{Z}, \tilde{\ell} \in \mathbb{Z}^d} \langle f, U_S D_A^n T_{\tilde{\ell}} \psi_A \rangle U_S D_A^n T_{\tilde{\ell}} \psi_A \\ &= \sum_{n \in \mathbb{Z}, \tilde{\ell} \in \mathbb{Z}^d} \langle f, D_B^n T_{S^{-1}\tilde{\ell}} \eta_B \rangle D_B^n T_{S^{-1}\tilde{\ell}} \eta_B \end{aligned}$$

$$= \sum_{n \in \mathbb{Z}, \vec{\ell} \in S^{-1}\mathbb{Z}^d} \langle f, D_B^n T_{\vec{\ell}} \eta_B \rangle D_B^n T_{\vec{\ell}} \eta_B.$$

Since S is an integral matrix with $|\det(S)| = 1$, we have $\mathbb{Z}^d = S\mathbb{Z}^d = S^{-1}\mathbb{Z}^d$. So we have

$$f = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d} \langle f, D_B^n T_{\vec{\ell}} \eta_B \rangle D_B^n T_{\vec{\ell}} \eta_B.$$

□

PROOF OF PROPOSITION 6.1.

PROOF. We have

$$\begin{aligned} |m_0(\vec{t})|^2 + |m_0(\vec{t} + \pi \vec{q}_A)|^2 &= \frac{1}{2} \left| \sum_{\vec{m} \in \mathbb{Z}^d} h_{\vec{m}} e^{-i\vec{m} \circ \vec{t}} \right|^2 + \frac{1}{2} \left| \sum_{\vec{m} \in \mathbb{Z}^d} h_{\vec{m}} e^{-i\vec{m} \circ (\vec{t} + \pi \cdot \vec{q}_A)} \right|^2 \\ &= \frac{1}{2} \left[\sum_{\vec{m} \in \mathbb{Z}^d, \vec{n} \in \mathbb{Z}^d} h_{\vec{m}} \overline{h_{\vec{n}}} e^{-i(\vec{m} - \vec{n}) \circ \vec{t}} + \sum_{\vec{m} \in \mathbb{Z}^d, \vec{n} \in \mathbb{Z}^d} (-1)^{(\vec{m} - \vec{n}) \circ \vec{q}_A} h_{\vec{m}} \overline{h_{\vec{n}}} e^{-i(\vec{m} - \vec{n}) \circ \vec{t}} \right] \\ &= \frac{1}{2} \left[\sum_{\vec{m} \in \mathbb{Z}^d, \vec{k} \in \mathbb{Z}^d} h_{\vec{m}} \overline{h_{\vec{m} + \vec{k}}} e^{i\vec{k} \circ \vec{t}} + \sum_{\vec{m} \in \mathbb{Z}^d, \vec{k} \in \mathbb{Z}^d} (-1)^{-\vec{k} \circ \vec{q}_A} h_{\vec{m}} \overline{h_{\vec{m} + \vec{k}}} e^{i\vec{k} \circ \vec{t}} \right] \end{aligned}$$

Here $\vec{k} = \vec{m} - \vec{n}$.

By property (4.3) in Partition Theorem, $\vec{k} \circ \vec{q}_A$ is odd when $\vec{k} \in (\ell_A + A\mathbb{Z}^d)$. Terms $(-1)^{-\vec{k} \circ \vec{q}_A} h_{\vec{m}} \overline{h_{\vec{m} + \vec{k}}} e^{i\vec{k} \circ \vec{t}}$ in the second sum cancel terms $h_{\vec{m}} \overline{h_{\vec{m} + \vec{k}}} e^{i\vec{k} \circ \vec{t}}$ in the first sum. The term $\vec{k} \circ \vec{q}_A$ is even when $\vec{k} \in A\mathbb{Z}^d$. So by Lawton's equations (6.1) we have

$$\begin{aligned} |m_0(\vec{t})|^2 + |m_0(\vec{t} + \pi \vec{q}_A)|^2 &= \sum_{\vec{m} \in \mathbb{Z}^d, \vec{k} \in A\mathbb{Z}^d} h_{\vec{m}} \overline{h_{\vec{m} + \vec{k}}} e^{i\vec{k} \circ \vec{t}} \\ &= \sum_{\vec{k} \in A\mathbb{Z}^d} \left(\sum_{\vec{m} \in \mathbb{Z}^d} h_{\vec{m}} \overline{h_{\vec{m} + \vec{k}}} \right) e^{i\vec{k} \circ \vec{t}} = \sum_{\vec{k} \in A\mathbb{Z}^d} \delta_{\vec{0}\vec{k}} e^{i\vec{k} \circ \vec{t}} = 1. \end{aligned}$$

□

PROOF OF LEMMA 8.1.

PROOF. For $z \in \mathbb{C}$, define

$$(12.2) \quad v(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

The function $v(z)$ is an entire function on \mathbb{C} .

By definition of m_0 we have

$$\begin{aligned} |d_j(\vec{\xi})| &= |m_0((A^\tau)^{-j} \vec{\xi}) - 1| \\ &= \left| \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} e^{-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi}} - 1 \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} (e^{-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi}} - 1) \right| \\
&= \left| \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} h_{\vec{n}} v(-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi}) [-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi}] \right| \\
&\leq \frac{1}{\sqrt{2}} \sum_{\vec{n} \in \Lambda_0} |h_n| |v(-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi})| \cdot |-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi}|.
\end{aligned}$$

The above estimate on $|d_j(\vec{\xi})|$ is bounded by $C_\Omega \cdot \frac{1}{\beta^j}$ for some constant C_Ω by the following obvious facts combined.

- (1) $|h_n| \leq 1$.
- (2) For each $n \in \Lambda_0$, the function $|v(-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi})|$ is continuous and assume its maximum on $\vec{\xi} \in \Omega$. Since Λ_0 is a finite set, $\max\{|v(-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi})|, \vec{\xi} \in \Omega, \vec{n} \in \Lambda_0\}$ is a finite number.
- (3) For each $\vec{n} \in \Lambda_0, \vec{\xi} \in \Omega$,

$$|-i\vec{n} \circ (A^\tau)^{-j} \vec{\xi}| \leq \max\{|\vec{n}| \cdot |\vec{\xi}|, \vec{\xi} \in \Omega \text{ and } \vec{n} \in \Lambda_0\} \cdot \frac{1}{\beta^j},$$

where $\max\{|\vec{n}| \cdot |\vec{\xi}|, \vec{\xi} \in \Omega \text{ and } \vec{n} \in \Lambda_0\}$ is a finite number.

□

PROOF OF LEMMA 8.3.

PROOF. Let $j \in \mathbb{N}, \vec{\xi} \in \mathbb{C}^d$ and $(A^\tau)^{-j} \vec{\xi} = \begin{pmatrix} \xi_1 \\ \dots \\ \xi_d \end{pmatrix} \in \mathbb{C}^d$. Define $\vec{\ell}_{\vec{\xi}} = \begin{pmatrix} \ell_1 \\ \dots \\ \ell_d \end{pmatrix} \in \mathbb{Z}^d$ by

$$\vec{\ell}_m = \begin{cases} -N_0, & \text{if } \Im(\xi_m) \leq 0; \\ N_0, & \text{if } \Im(\xi_m) > 0. \end{cases} \quad m = 1, 2, \dots, d.$$

then $\Im((\vec{n} - \vec{\ell}_{\vec{\xi}}) \circ ((A^\tau)^{-j} \vec{\xi})) \leq 0$ for $\vec{n} \in \Lambda_0$. Denote $B_0 \equiv 2\sqrt{d}N_0$. It is clear that $|\vec{\ell}_{\vec{\xi}}| \leq B_0$ and $|\vec{n} - \vec{\ell}_{\vec{\xi}}| \leq B_0, \forall \vec{n} \in \Lambda_0$. By inequality (8.4) we have

$$|e^{-i(\vec{n} - \vec{\ell}_{\vec{\xi}}) \circ ((A^\tau)^{-j} \vec{\xi})} - 1| \leq \min(2, |(\vec{n} - \vec{\ell}_{\vec{\xi}}) \circ ((A^\tau)^{-j} \vec{\xi})|), \forall \vec{n} \in \Lambda_0$$

So, we have

$$(12.3) \quad |e^{-i(\vec{n} - \vec{\ell}_{\vec{\xi}}) \circ ((A^\tau)^{-j} \vec{\xi})} - 1| \leq \min\left(2, \frac{B_0 |\vec{\xi}|}{\beta^j}\right), \forall \vec{n} \in \Lambda_0.$$

We also have

$$\begin{aligned}
|e^{-i\vec{\ell}_{\vec{\xi}} \circ ((A^\tau)^{-j} \vec{\xi})}| &= e^{\vec{\ell}_{\vec{\xi}} \circ ((A^\tau)^{-j} \Im(\vec{\xi}))} \\
&\leq e^{|\vec{\ell}_{\vec{\xi}}| \|(A^\tau)^{-1}\|^j |\Im(\vec{\xi})|}.
\end{aligned}$$

This implies

$$(12.4) \quad |e^{-i\vec{\ell}_{\xi} \circ ((A^\tau)^{-j}\vec{\xi})}| \leq \exp\left(\frac{B_0|\Im(\vec{\xi})|}{\beta^j}\right).$$

Since

$$m_0((A^\tau)^{-j}\vec{\xi}) = e^{-i\vec{\ell}_{\xi} \circ ((A^\tau)^{-j}\vec{\xi})} \left(1 + \sum_{\vec{n} \in \Lambda_0} \frac{1}{\sqrt{2}} h_{\vec{n}} (e^{-i(\vec{n}-\vec{\ell}_{\xi}) \circ ((A^\tau)^{-j}\vec{\xi})} - 1)\right),$$

by (12.3) and (12.4) we obtain

$$\begin{aligned} |m_0((A^\tau)^{-j}\vec{\xi})| &\leq |e^{-i\vec{\ell}_{\xi} \circ ((A^\tau)^{-j}\vec{\xi})}| \cdot \left(1 + \sum_{\vec{n} \in \Lambda_0} \frac{1}{\sqrt{2}} |h_{\vec{n}}| \cdot |e^{-i(\vec{n}-\vec{\ell}_{\xi}) \circ ((A^\tau)^{-j}\vec{\xi})} - 1|\right) \\ &\leq \exp\left(\frac{B_0|\Im(\vec{\xi})|}{\beta^j}\right) \left(1 + \frac{1}{\sqrt{2}} (2N_0 + 1)^d \min(2, \frac{B_0|\vec{\xi}|}{\beta^j})\right) \\ &\leq (1 + C_0 \min(1, \frac{|\vec{\xi}|}{\beta^j})) \exp\left(\frac{B_0|\Im(\vec{\xi})|}{\beta^j}\right). \end{aligned}$$

where $C_0 \equiv \max(\sqrt{2}(2N_0 + 1)^d, \frac{B_0}{\sqrt{2}}(2N_0 + 1)^d)$.

□

PROOF OF PROPOSITION 8.2.

PROOF. By Schwartz's Paley-Wiener Theorem, it suffices to prove that the function g satisfies the inequality (8.1).

Let $\vec{\xi} \in \mathbb{R}^d$. We assume $\vec{\xi} \neq \vec{0}$. (The case $\vec{\xi} = \vec{0}$ is trivial.) Denote $B \equiv \sum_{j=1}^{\infty} \frac{B_0}{\beta^j}$. By Lemma 8.3 we have

$$\begin{aligned} |g(\vec{\xi})| &= \left| \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^{\infty} m_0((A^\tau)^{-j}\vec{\xi}) \right| \\ &\leq \frac{1}{(2\pi)^{d/2}} e^{B|\Im(\vec{\xi})|} \prod_{j=1}^{\infty} (1 + C_0 \min(1, \frac{|\vec{\xi}|}{\beta^j})). \end{aligned}$$

On the other hand, the sequence $\{\beta^j\}$ is monotonically increasing to $+\infty$. Let $I_j \equiv [\beta^j, \beta^{j+1})$, $j \in \mathbb{N}$ and $I_0 \equiv (0, \beta)$. The set of intervals $\{I_j, j \geq 0\}$ is a partition of $(0, \infty)$. So $|\vec{\xi}| \in I_{j_0}$ for some integer $j_0 \geq 0$. We have

$$\begin{aligned} (1 + C_0)^{j_0} &= (\beta^{j_0})^{\log_{\beta}(1+C_0)} \\ &\leq |\vec{\xi}|^{\log_{\beta}(1+C_0)} \\ &\leq (1 + |\vec{\xi}|)^N, \end{aligned}$$

where N is the smallest natural number no less than $\log_{\beta}(1+C_0)$. This is a constant related to A and N_0 only. So, we have

$$\begin{aligned} |g(\vec{\xi})| &\leq \frac{1}{(2\pi)^{d/2}} (1 + C_0)^{j_0} e^{B|\Im(\vec{\xi})|} \prod_{j=j_0+1}^{\infty} (1 + C_0 \min(1, \frac{|\vec{\xi}|}{\beta^j})) \\ &\leq (1 + |\vec{\xi}|)^N e^{B|\Im(\vec{\xi})|} \cdot \left(\frac{1}{(2\pi)^{d/2}} \prod_{j=j_0+1}^{\infty} (1 + C_0 \min(1, \frac{|\vec{\xi}|}{\beta^j})) \right). \end{aligned}$$

Since $|\vec{\xi}| \in I_{j_0} = [\beta^{j_0}, \beta^{j_0+1})$, $\frac{|\vec{\xi}|}{\beta^{j_0+1}} < 1$. We have

$$\begin{aligned}
& \frac{1}{(2\pi)^{d/2}} \prod_{j=j_0+1}^{\infty} (1 + C_0 \min(1, \frac{|\vec{\xi}|}{\beta^j})) \\
&= \frac{1}{(2\pi)^{d/2}} \prod_{j=j_0+1}^{\infty} (1 + C_0 \frac{|\vec{\xi}|}{\beta^{j_0+1}} \cdot \frac{1}{\beta^{j-(j_0+1)}}) \\
&\leq \frac{1}{(2\pi)^{d/2}} \prod_{k=0}^{\infty} (1 + \frac{C_0}{\beta^k}) \\
&\leq \frac{1}{(2\pi)^{d/2}} e^{\sum \frac{C_0}{\beta^k}}.
\end{aligned}$$

Denote $C \equiv \frac{1}{(2\pi)^{d/2}} e^{\sum \frac{C_0}{\beta^k}}$. This is a constant decided by the matrix A .

Combining the above argument, we have the desired inequality

$$|g(\vec{\xi})| \leq C(1 + |\vec{\xi}|)^N e^{B|\Im(\vec{\xi})|}.$$

□

PROOF OF PROPOSITION 9.1.

PROOF. By the equations (9.1) and (5.3) $T_{\vec{k}} D_A = D_A T_{A\vec{k}}$, we have

$$\begin{aligned}
I_{-1} &= \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^{-1} T_{\vec{k}} \varphi \rangle D_A^{-1} T_{\vec{k}} \varphi \\
&= \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^{-1} T_{\vec{k}} \sum_{\vec{p} \in \mathbb{Z}^d} h_{\vec{p}} D_A T_{\vec{p}} \varphi \rangle D_A^{-1} T_{\vec{k}} \sum_{\vec{q} \in \mathbb{Z}^d} h_{\vec{q}} D_A T_{\vec{q}} \varphi \\
&= \sum_{\vec{p} \in \mathbb{Z}^d} \sum_{\vec{q} \in \mathbb{Z}^d} \sum_{\vec{k} \in \mathbb{Z}^d} \overline{h_{\vec{p}}} h_{\vec{q}} \langle f, T_{\vec{p}+A\vec{k}} \varphi \rangle T_{\vec{q}+A\vec{k}} \varphi \\
&= \sum_{\vec{p} \in \mathbb{Z}^d} \sum_{\vec{q} \in \mathbb{Z}^d} \sum_{\vec{k} \in A\mathbb{Z}^d} \overline{h_{\vec{p}}} h_{\vec{q}} \langle f, T_{\vec{p}+\vec{k}} \varphi \rangle T_{\vec{q}+\vec{k}} \varphi. \\
F_{-1} &= \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^{-1} T_{\vec{k}} \psi \rangle D_A^{-1} T_{\vec{k}} \psi \\
&= \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^{-1} T_{\vec{k}} \sum_{\vec{p} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{p}} \overline{h_{\vec{\ell}_A - \vec{p}}} D_A T_{\vec{p}} \varphi \rangle D_A^{-1} T_{\vec{k}} \sum_{\vec{q} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ \vec{q}} \overline{h_{\vec{\ell}_A - \vec{q}}} D_A T_{\vec{q}} \varphi \\
&= \sum_{\vec{p} \in \mathbb{Z}^d} \sum_{\vec{q} \in \mathbb{Z}^d} \sum_{\vec{k} \in \mathbb{Z}^d} (-1)^{\vec{q}_A \circ (\vec{p} + \vec{q})} h_{\vec{\ell}_A - \vec{p}} \overline{h_{\vec{\ell}_A - \vec{q}}} \langle f, T_{\vec{p}+A\vec{k}} \varphi \rangle T_{\vec{q}+A\vec{k}} \varphi \\
&= \sum_{\vec{p} \in \mathbb{Z}^d} \sum_{\vec{q} \in \mathbb{Z}^d} \sum_{\vec{k} \in A\mathbb{Z}^d} (-1)^{\vec{q}_A \circ (\vec{p} + \vec{q})} h_{\vec{\ell}_A - \vec{p}} \overline{h_{\vec{\ell}_A - \vec{q}}} \langle f, T_{\vec{p}+\vec{k}} \varphi \rangle T_{\vec{q}+\vec{k}} \varphi.
\end{aligned}$$

Use substitutions $\vec{n} \equiv \vec{q} + \vec{k}$ and $\vec{m} \equiv \vec{p} - \vec{q}$, we have $\vec{q} \equiv \vec{n} - \vec{k}$ and $\vec{p} \equiv \vec{m} + \vec{n} - \vec{k}$, then

$$I_{-1} = \sum_{(\vec{m}, \vec{n}) \in \mathbb{Z}^d \times \mathbb{Z}^d} \left(\sum_{\vec{k} \in A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{n} - \vec{k}}} h_{\vec{n} - \vec{k}} \right) \langle f, T_{\vec{m} + \vec{n}} \varphi \rangle T_{\vec{n}} \varphi.$$

$$\begin{aligned}
F_{-1} &= \sum_{(\vec{m}, \vec{n}) \in \mathbb{Z}^d \times \mathbb{Z}^d} \left(\sum_{\vec{k} \in A\mathbb{Z}^d} (-1)^{q_{\vec{A}} \circ (\vec{m} + 2\vec{n} - 2\vec{k})} h_{\vec{\ell}_A - \vec{m} - \vec{n} + \vec{k}} \overline{h_{\vec{\ell}_A - \vec{n} + \vec{k}}} \right) \langle f, T_{\vec{m} + \vec{n}} \varphi \rangle T_{\vec{n}} \varphi \\
&= \sum_{(\vec{m}, \vec{n}) \in \mathbb{Z}^d \times \mathbb{Z}^d} \left(\sum_{\vec{k} \in A\mathbb{Z}^d} (-1)^{q_{\vec{A}} \circ \vec{m}} h_{\vec{\ell}_A - \vec{m} - \vec{n} + \vec{k}} \overline{h_{\vec{\ell}_A - \vec{n} + \vec{k}}} \right) \langle f, T_{\vec{m} + \vec{n}} \varphi \rangle T_{\vec{n}} \varphi.
\end{aligned}$$

Denote

$$\begin{aligned}
\alpha_{\vec{m}, \vec{n}} &\equiv \sum_{\vec{k} \in A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{n} - \vec{k}}} h_{\vec{n} - \vec{k}}, \\
\beta_{\vec{m}, \vec{n}} &\equiv \sum_{\vec{k} \in A\mathbb{Z}^d} (-1)^{q_{\vec{A}} \circ \vec{m}} h_{\vec{\ell}_A - \vec{m} - \vec{n} + \vec{k}} \overline{h_{\vec{\ell}_A - \vec{n} + \vec{k}}}.
\end{aligned}$$

We have

$$(12.5) \quad I_{-1} + F_{-1} = \sum_{(\vec{m}, \vec{n}) \in \mathbb{Z}^d \times \mathbb{Z}^d} (\alpha_{\vec{m}, \vec{n}} + \beta_{\vec{m}, \vec{n}}) \langle f, T_{\vec{m} + \vec{n}} \varphi \rangle T_{\vec{n}} \varphi.$$

By Partition Theorem, Equations (4.1) and (4.3), we have

$$(12.6) \quad (-1)^{q_{\vec{A}} \circ \vec{m}} = \begin{cases} 1, & \vec{m} \in A\mathbb{Z}^d, \\ -1, & \vec{m} \in \vec{\ell}_A + A\mathbb{Z}^d. \end{cases}$$

For $\vec{m} \in A\mathbb{Z}^d$, by (12.6) and Partition Theorem Equation (4.4) we have

$$\begin{aligned}
\alpha_{\vec{m}, \vec{n}} + \beta_{\vec{m}, \vec{n}} &= \sum_{\vec{k} \in A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{n} - \vec{k}}} h_{\vec{n} - \vec{k}} + \sum_{\vec{k} \in A\mathbb{Z}^d} h_{\vec{\ell}_A - \vec{m} - \vec{n} + \vec{k}} \overline{h_{\vec{\ell}_A - \vec{n} + \vec{k}}} \\
&= \sum_{\vec{\ell} \in \vec{n} - A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{\ell}}} h_{\vec{\ell}} + \sum_{\vec{\ell} \in \vec{\ell}_A - \vec{m} - \vec{n} + A\mathbb{Z}^d} h_{\vec{\ell}} \overline{h_{\vec{m} + \vec{\ell}}} \\
&= \sum_{\vec{\ell} \in \mathbb{Z}^d} \overline{h_{\vec{m} + \vec{\ell}}} h_{\vec{\ell}}.
\end{aligned}$$

Then by Lawton's equations (6.1) we obtain

$$(12.7) \quad \alpha_{\vec{m}, \vec{n}} + \beta_{\vec{m}, \vec{n}} = \delta_{\vec{m}, \vec{0}}, \quad \vec{m} \in A\mathbb{Z}^d.$$

For $\vec{m} \in \vec{\ell}_A + A\mathbb{Z}^d$, by (12.6) and Partition Theorem Equation (4.5) we have

$$\begin{aligned}
\alpha_{\vec{m}, \vec{n}} + \beta_{\vec{m}, \vec{n}} &= \sum_{\vec{k} \in A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{n} - \vec{k}}} h_{\vec{n} - \vec{k}} - \sum_{\vec{k} \in A\mathbb{Z}^d} h_{\vec{\ell}_A - \vec{m} - \vec{n} + \vec{k}} \overline{h_{\vec{\ell}_A - \vec{n} + \vec{k}}} \\
&= \sum_{\vec{\ell} \in \vec{n} - A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{\ell}}} h_{\vec{\ell}} - \sum_{\vec{\ell} \in \vec{\ell}_A - \vec{m} - \vec{n} + A\mathbb{Z}^d} h_{\vec{\ell}} \overline{h_{\vec{m} + \vec{\ell}}} \\
&= \sum_{\vec{\ell} \in \vec{n} - A\mathbb{Z}^d} \overline{h_{\vec{m} + \vec{\ell}}} h_{\vec{\ell}} - \sum_{\vec{\ell} \in \vec{n} - A\mathbb{Z}^d} h_{\vec{\ell}} \overline{h_{\vec{m} + \vec{\ell}}} \\
&= 0.
\end{aligned}$$

Or

$$(12.8) \quad \alpha_{\vec{m}, \vec{n}} + \beta_{\vec{m}, \vec{n}} = 0, \quad \vec{m} \in \vec{\ell}_A + A\mathbb{Z}^d.$$

Combine (12.5), (12.7) and (12.8) we obtain

$$I_{-1} + F_{-1} = \sum_{\vec{n} \in \mathbb{Z}^d} \langle f, T_{\vec{n}} \varphi \rangle T_{\vec{n}} \varphi.$$

This is

$$(12.9) \quad \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^{-1} T_{\vec{k}} \varphi \rangle D_A^{-1} T_{\vec{k}} \varphi + \sum_{\vec{k} \in \mathbb{Z}^d} \langle f, D_A^{-1} T_{\vec{k}} \psi \rangle D_A^{-1} T_{\vec{k}} \psi = \sum_{\vec{n} \in \mathbb{Z}^d} \langle f, T_{\vec{n}} \varphi \rangle T_{\vec{n}} \varphi.$$

After performing the following operations on equation (12.9) we will have the desired Equation (9.3) in this Proposition:

- (1) Replace f by $D^{-J-1}f$ in (12.9) and then.
- (2) Apply D_A^{J+1} to the two sides.
- (3) Use the facts that $\langle D_A^{-J-1}f, D_A^{-1}T_{\vec{k}}\varphi \rangle = \langle f, D_A^J T_{\vec{k}}\varphi \rangle$, $\langle D_A^{-J-1}f, D_A^{-1}T_{\vec{k}}\psi \rangle = \langle f, D_A^J T_{\vec{k}}\psi \rangle$ and $\langle D_A^{-J-1}f, T_{\vec{k}}\varphi \rangle = \langle f, D_A^{J+1}T_{\vec{k}}\varphi \rangle$.

□

PROOF OF PROPOSITION 9.2.

PROOF. (1) By Proposition 8.2 the scaling function φ has a compact support. Let B be a natural number such that the set $[-B, B]^d$ contains the support of φ . We will write $E_0 \equiv [-\frac{1}{2}, \frac{1}{2}]^d$, $E_B \equiv [-B - \frac{1}{2}, B + \frac{1}{2}]^d$ and $\Lambda_B \equiv \mathbb{Z}^d \cap [-B, B]^d$. For $\vec{n} \in \mathbb{Z}^d$, we have $\vec{n} = (2B+1)\vec{\ell} + \vec{d}$, $\vec{\ell} \in \mathbb{Z}^d$, $\vec{d} \in \Lambda_B$. Here $\vec{\ell}$ and $\vec{d} \in \Lambda_B$ are uniquely determined by \vec{n} . We will denote $E_{\vec{\ell}, \vec{d}} \equiv E_B + (2B+1)\vec{\ell} + \vec{d}$. We have the following partitions.

$$\begin{aligned} \mathbb{Z}^d &= \bigcup_{\vec{d} \in \Lambda_B} \bigcup_{\vec{\ell} \in \mathbb{Z}^d} ((2B+1)\vec{\ell} + \vec{d}). \\ \mathbb{R}^d &= \bigcup_{\vec{\ell} \in \mathbb{Z}^d} E_{\vec{\ell}, \vec{d}}, \text{ for each fixed } \vec{d} \in \Lambda_B. \end{aligned}$$

So, for a fixed $\vec{d} \in \Lambda_B$ the functions $\chi_{E_{\vec{\ell}, \vec{d}}} f$, $\vec{\ell} \in \mathbb{Z}^d$ have disjoint supports and sum to f , hence are orthogonal. Then we have

$$\begin{aligned} L_0(f) &= \sum_{\vec{d} \in \Lambda_B} \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, T_{(2B+1)\vec{\ell} + \vec{d}} \varphi \rangle|^2 = \sum_{\vec{d} \in \Lambda_B} \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle \chi_{E_{\vec{\ell}, \vec{d}}} f, T_{(2B+1)\vec{\ell} + \vec{d}} \varphi \rangle|^2 \\ &\leq \sum_{\vec{d} \in \Lambda_B} \sum_{\vec{\ell} \in \mathbb{Z}^d} \|\chi_{E_{\vec{\ell}, \vec{d}}} f\|^2 \cdot \|\varphi\|^2 = (2B+1)^2 \|f\|^2 \|\varphi\|^2. \end{aligned}$$

So we have

$$L_J(f) = \sum_{\vec{\ell} \in \mathbb{Z}^d} | \langle (D_A^J)^* f, T_{\vec{\ell}} \varphi \rangle |^2 \leq (2B+1)^2 \|\varphi\|^2 \| (D_A^J)^* f \|^2.$$

This implies

$$(12.10) \quad L_J(f) \leq (2B+1)^2 \|\varphi\|^2 \|f\|^2, \forall J \in \mathbb{Z}$$

and

$$(12.11) \quad \lim_{\rho \rightarrow \infty} \limsup_{J \rightarrow +\infty} L_J(f_{\rho}) = 0.$$

(2) We have

$$L_{-J}(f) = \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^{-J} T_{\vec{\ell}} \varphi \rangle|^2$$

$$\begin{aligned}
&= \sum_{\vec{d} \in \Lambda_B} \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}} \vec{\varphi} \rangle|^2 \\
&= \sum_{\vec{d} \in \Lambda_B} \sum_{\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}} \vec{\varphi} \rangle|^2 + \sum_{\vec{d} \in \Lambda_B} |\langle f, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle|^2.
\end{aligned}$$

It is clear that $E_0 \subset E_B + \vec{d}$ and $(E_B + (2B+1)\vec{\ell} + \vec{d}) \cap E_0 = \emptyset, \forall \vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}$. The support of the function $D_A^{-J} T_{\vec{\ell}} \vec{\varphi}$ is contained in $A^J(E_B + \vec{\ell})$. We have

$$\begin{aligned}
&\sum_{\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}} \vec{\varphi} \rangle|^2 \\
&= \sum_{\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\langle \chi_{A^J(E_B + (2B+1)\vec{\ell} + \vec{d})} f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}} \vec{\varphi} \rangle|^2 \\
&\leq \|\phi\|^2 \cdot \sum_{\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}} \int_{A^J(E_B + (2B+1)\vec{\ell} + \vec{d})} |f|^2 d\mu \\
&\leq \|\varphi\|^2 \cdot \int_{\mathbb{R}^d \setminus A^J E_0} |f|^2 d\mu.
\end{aligned}$$

Since A is expansive, $\lim_{J \rightarrow +\infty} A^J E_0 = \mathbb{R}^d$, $\lim_{J \rightarrow +\infty} \int_{\mathbb{R}^d \setminus A^J E_0} |f|^2 d\mu = 0$. So

$$\lim_{J \rightarrow +\infty} \sum_{\vec{d} \in \Lambda_B} \sum_{\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\langle f, D_A^{-J} T_{(2B+1)\vec{\ell}+\vec{d}} \vec{\varphi} \rangle|^2 = 0.$$

(3) To complete the proof of this Proposition, we need to show that

$$(12.12) \quad \lim_{J \rightarrow +\infty} \sum_{\vec{d} \in \Lambda_B} |\langle f, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle|^2 = 0.$$

Let $f_N \equiv \chi_{[-N, N]^2} \cdot f$. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ be large such that $\|f - f_N\| \leq \frac{\varepsilon}{2\|\varphi\|}$. Then we have $|\langle f, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle| \leq |\langle f_N, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle| + \frac{\varepsilon}{2}$. Since

$$\begin{aligned}
|\langle f_N, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle| &= |\langle D_A^J f_N, T_{\vec{d}} \vec{\varphi} \rangle| \\
&= |\langle \chi_{A^{-J}[-N, N]^2} D_A^J f_N, T_{\vec{d}} \vec{\varphi} \rangle| \\
&= |\langle D_A^J f_N, \chi_{A^{-J}[-N, N]^2} T_{\vec{d}} \vec{\varphi} \rangle|,
\end{aligned}$$

we have

$$|\langle f_N, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle| \leq \|D_A^J f_N\| \cdot \sqrt{\int_{A^{-J}[-N, N]^d} |T_{\vec{d}} \vec{\varphi}|^2 d\mu}.$$

since $\lim_{J \rightarrow \infty} \mu(A^{-J}[-N, N]^d) = 0$ and $\|D_A^J f_N\| \leq \|f\|$, so $\lim_{J \rightarrow +\infty} |\langle f, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle|^2 = 0$ for each $\vec{d} \in \Lambda_B$. Since Λ_B is a finite set, we have

$$\lim_{J \rightarrow +\infty} \sum_{\vec{d} \in \Lambda_B} |\langle f, D_A^{-J} T_{\vec{d}} \vec{\varphi} \rangle|^2 = 0.$$

□

PROOF OF PROPOSITION 9.3

PROOF. We denote

$$\begin{aligned} U_J(f) &\equiv (2\pi)^d \int_{\mathbb{R}^d} |\widehat{f}(\vec{s})|^2 |\widehat{\varphi}((A^\tau)^{-J}\vec{s})|^2 d\vec{s}, \\ V_J(f) &\equiv (2\pi)^d \int_{\mathbb{R}^d} \sum_{\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}} \left(\widehat{f}(\vec{s}) \cdot \overline{\widehat{f}(\vec{s} - 2\pi(A^\tau)^J\vec{\ell})} \cdot \widehat{\varphi}((A^\tau)^{-J}\vec{s} - 2\pi\vec{\ell}) \cdot \overline{\widehat{\varphi}((A^\tau)^{-J}\vec{s})} \right) d\vec{s}. \end{aligned}$$

By Lemma 9.2, we have $L_J(f) = U_J(f) + V_J(f)$. It is enough to prove that

$$(12.13) \quad \lim_{J \rightarrow +\infty} U_J(f) = \|f\|^2$$

and

$$(12.14) \quad \lim_{J \rightarrow +\infty} V_J(f) = 0.$$

1. Recall that A^τ is also expansive, so $\lim_{J \rightarrow +\infty} (A^\tau)^{-J}\vec{s} = \vec{0}, \forall \vec{s} \in \mathbb{R}^d$. Also, by definition (7.1), Corollary 6.1 and Proposition 8.1, g is bounded and continuous on \mathbb{R}^d . Also, $(2\pi)^{\frac{d}{2}}\widehat{\varphi}(\vec{0}) = (2\pi)^{\frac{d}{2}}g(\vec{0}) = 1$. By Lebesgue Dominate Convergence Theorem we have

$$\begin{aligned} \lim_{J \rightarrow +\infty} U_J(f) &= \lim_{J \rightarrow +\infty} (2\pi)^d \int_{\mathbb{R}^d} |\widehat{f}(\vec{s})|^2 \cdot |\widehat{\varphi}((A^\tau)^{-J}\vec{s})|^2 d\vec{s} \\ &= \lim_{J \rightarrow +\infty} \int_{\mathbb{R}^d} |\widehat{f}(\vec{s})|^2 \cdot |(2\pi)^{\frac{d}{2}}g((A^\tau)^{-J}\vec{s})|^2 d\vec{s} \\ &= \|\widehat{f}\|^2 = \|f\|^2. \end{aligned}$$

This proves (12.13).

2. Let $\rho \in \mathbb{R}^+$, Δ_ρ be the open ball with center $\vec{0}$ and radius ρ . Since A^τ is expansive, $\beta \equiv \|(A^\tau)^{-1}\|^{-1} > 1$. Denote $a \equiv \log_\beta(2\rho)$. Let J_ρ be the smallest natural number in the interval $(a, +\infty)$. When $J \geq J_\rho$, $(A^\tau)^J\Delta_1$ contains an open ball $\Delta_{2\rho}$. Since $\Delta_1 \cap \mathbb{Z}^d = \{\vec{0}\}$, $2\pi(A^\tau)^J\Delta_1 \cap 2\pi(A^\tau)^J\mathbb{Z}^d = \{\vec{0}\}$. Also we have $\Delta_{2\rho} \subseteq (A^\tau)^J\Delta_1 \subseteq 2\pi(A^\tau)^J\Delta_1$. These facts implies that when $J \geq J_\rho$ the distance between $\vec{0}$ and $2\pi(A^\tau)^J\vec{\ell}$ is greater than 2ρ . So for each $\vec{\ell} \in \mathbb{Z}^d \setminus \{\vec{0}\}$, the support of $\widehat{f}_\rho(\vec{t})$ which is Δ_ρ and the support of $\widehat{f}_\rho(\vec{t} - 2\pi(A^\tau)^J\vec{\ell})$ which is $\Delta_\rho + 2\pi(A^\tau)^J\vec{\ell}$ are disjoint. This implies that the product $\widehat{f}_\rho(\vec{t})\overline{\widehat{f}_\rho(\vec{t} - 2\pi(A^\tau)^J\vec{\ell})} \equiv 0$ when $J \geq J_\rho$. Therefore, we have

$$\lim_{J \rightarrow +\infty} V_J(f_\rho) = 0, \forall \rho \in \mathbb{R}^+.$$

Together with (12.13), we have proved that

$$(12.15) \quad \lim_{J \rightarrow +\infty} L_J(f_\rho) = \|f_\rho\|^2, \forall f \in L^2(\mathbb{R}^d), \forall \rho \in \mathbb{R}^+.$$

3. Let $D_\rho \equiv \sum_{\vec{\ell} \in \mathbb{Z}^d} (\langle f_\rho, D_A^J T_{\vec{\ell}} \varphi \rangle \overline{\langle f_{\vec{\rho}}, D_A^J T_{\vec{\ell}} \varphi \rangle} + \langle f_{\vec{\rho}}, D_A^J T_{\vec{\ell}} \varphi \rangle \overline{\langle f_\rho, D_A^J T_{\vec{\ell}} \varphi \rangle})$. Then

$$\begin{aligned} |D_\rho| &\leq 2 \sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f_\rho, D_A^J T_{\vec{\ell}} \varphi \rangle| \cdot |\langle f_{\vec{\rho}}, D_A^J T_{\vec{\ell}} \varphi \rangle| \\ &\leq 2 \sqrt{\sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f_\rho, D_A^J T_{\vec{\ell}} \varphi \rangle|^2} \cdot \sqrt{\sum_{\vec{\ell} \in \mathbb{Z}^d} |\langle f_{\vec{\rho}}, D_A^J T_{\vec{\ell}} \varphi \rangle|^2} \\ &= 2\sqrt{L_J(f_\rho)} \cdot \sqrt{L_J(f_{\vec{\rho}})}. \end{aligned}$$

By (12.10), we have

$$(12.16) \quad |D_\rho| \leq 2(2B+1)^2 \|\varphi\|^2 \|f_\rho\| \|f_{\bar{\rho}}\|.$$

We have

$$\begin{aligned} L_J(f) - \|f\|^2 &= L_J(f_\rho + f_{\bar{\rho}}) - \|f\|^2 \\ &= \sum_{\bar{\ell} \in \mathbb{Z}^d} \langle f_\rho + f_{\bar{\rho}}, D_A^J T_{\bar{\ell}} \varphi \rangle \overline{\langle f_\rho + f_{\bar{\rho}}, D_A^J T_{\bar{\ell}} \varphi \rangle} - \|f\|^2 \\ &= L_J(f_\rho) - \|f\|^2 + L_J(f_{\bar{\rho}}) + D_\rho \\ &= (L_J(f_\rho) - \|f_\rho\|^2) - \|f_{\bar{\rho}}\|^2 + L_J(f_{\bar{\rho}}) + D_\rho. \end{aligned}$$

By (12.15), (12.11) and (12.16) we have

$$\limsup_{J \rightarrow +\infty} |L_J(f) - \|f\|^2| \leq 0 + \|f_{\bar{\rho}}\|^2 + 0 + 2(2B+1)^2 \|\varphi\|^2 \|f_\rho\| \|f_{\bar{\rho}}\|, \forall \rho \in \mathbb{R}^+.$$

Let $\rho \rightarrow +\infty$, we have (12.14)

$$\lim_{J \rightarrow \infty} L_J(f) = \|f\|^2.$$

□